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On some properties of (p, k)-Mittag-Leffler function

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Abstract

We aim to present a new generalization of Mittag-Leffler function. Then we investigate certain properties and formulas associated with newly introduced (p, k)-Mittag-Leffler function. Various results including representation in terms of generalized hypergeometric function, integral transforms like Laplace transform, Beta transform, Whittaker transform have been established and proved. Finally some interesting special cases are discussed.

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1. Introduction and Preliminaries

In this section, we present results and definitions known and important for the development of the following sections.

The Gauss hypergeometric function is defined as (see [1])

Definition 1. *Let* $|z| < 1,\ z, \delta_1, \delta_2 \in \mathbb{C},\ \delta_3 \in \mathbb{C} \setminus \mathbb{Z}_0^-,$ *then*

$$ {}_2F_1(\delta_1, \delta_2; \delta_3; z) = \sum_{n=0}^{\infty} \frac{(\delta_1)_n (\delta_2)_n}{(\delta_3)_n} \frac{z^n}{n!}, \tag{1.1}$$

where $(\delta_1)_n$ is familiar Pochhammar symbol (see [2]).

The series ${}_2F_1(\delta_1, \delta_2; \delta_3; z)$ is convergent in the following cases:

1. for $|z| < 1$; the series converges absolutely.
2. for $|z| = 1$; the series converges absolutely for $\Re(\delta_3 - \delta_1 - \delta_2) > 0$.
3. for $|z| = 1\ (z \neq 1)$; the series converges conditionally for $-1 < \Re(\delta_3 - \delta_1 - \delta_2) \leq 0$.

The integral form of the Gauss hypergeometric function is given as (see [1, 3]):

$$ {}_2F_1(\delta_1, \delta_2; \delta_3; z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)\Gamma(\delta_3 - \delta_2)} \int_0^1 t^{\delta_2 - 1}(1-t)^{\delta_3 - \delta_2 - 1}(1 - zt)^{-\delta_1} dt, \\ (|z| < 1; \Re(\delta_3) > \Re(\delta_2) > 0). \tag{1.2} $$

A more generalized form of the hypergeometric function is ${}_rF_s$, defined as (see [1, 3]):

Definition 2. *Let* $\delta_1, \cdots, \delta_r \in \mathbb{C}; \omega_1, \cdots, \omega_s \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then*

$$ {}_rF_s\left[\begin{array}{c} \delta_1, \cdots, \delta_r; \\ \omega_1, \cdots, \omega_s; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\delta_1)_n \dots (\delta_r)_n}{(\omega_1)_n \dots (\omega_s)_n} \frac{z^n}{n!}. \tag{1.3} $$

The series ${}_rF_s(.)$ is convergent in the following cases (see [1]):

1. Converges absolutely $\forall \ z \in \mathbb{C}$, if $r \leq s$.
2. (a) Converges absolutely for $|z| < 1$ and $r = s + 1$.
   (b) Diverges for $|z| > 1$ and $r = s + 1$.
3. Diverges for $z \neq 0$, if $r > s + 1$.
4. Absolutely convergent for $|z| = 1$, when $r = s + 1$ and

$$ \Re\left[\sum_{j=1}^{s} \omega_j - \sum_{i=1}^{r} \delta_i\right] > 0. $$

5. Conditionally convergent for $|z| = 1$ $(z \neq 1)$, if $r = s + 1$ and when

$$ -1 < \Re\left[\sum_{j=1}^{s} \omega_j - \sum_{i=1}^{r} \delta_i\right] \leq 0. $$

where $(\delta)_n$ is the Pochhammer symbol defined by (see [2])

$$ (\delta)_n = \begin{cases} 1 & (n = 0;\ \delta \in \mathbb{C} \setminus \{0\}) \\ \delta(\delta+1)(\delta+2)\dots(\delta+n-1) & (n \in \mathbb{N};\ \delta \in \mathbb{C}) \end{cases} \\ = \frac{\Gamma(\delta+n)}{\Gamma(\delta)} \qquad (n \in \mathbb{N};\ \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{1.4} $$

and $\Gamma(\delta)$ is the familiar Gamma function.

The Pochhammer k-symbol due to Diaz and Pariguan [4] is defined as

Definition 3. *Let* $\delta \in \mathbb{C}, k \in \mathbb{R}$ *and* $n \in \mathbb{N}$, *then the Pochhemmer* k*-symbol is defined as: (Diaz and Pariguan [4])*

$$ (\delta)_{n,k} = \delta(\delta + k)(\delta + 2k) \cdots (\delta + (n-1)k). \tag{1.5} $$

and k-Gamma function is defined as

Definition 4. *For $\delta \in \mathbb{C};\ \Re(\delta) > 0;\ k \in \mathbb{R}^+$, the k-gamma function is defined by [4]:*

$$\Gamma_k(\delta) = \int_0^\infty t^{\delta-1} e^{-\frac{t^k}{k}} dt. \tag{1.6}$$

Proposition 1. *The k-gamma function and Pochhammer k-symbol are related as (see [4]):*
For $n \in \mathbb{N};\ k \in \mathbb{R}^+$

$$(\delta)_{n,k} = \begin{cases} \dfrac{\Gamma_k(\delta + nk)}{\Gamma_k(\delta)} & (\delta \in \mathbb{C} \setminus \{0\}) \\ \delta(\delta + k) \cdots (\delta + (n-1)k), & (\delta \in \mathbb{C}). \end{cases} \tag{1.7}$$

For $\delta \in \mathbb{C},\ \Re(\delta) > 0;\ k \in \mathbb{R}^+$

$$\Gamma_k(\delta) = k^{\frac{\delta}{k}-1} \Gamma\left(\frac{\delta}{k}\right). \tag{1.8}$$

Recently, Gehlot [5] introduced a new modification of the k-Gamma function ${}_p\Gamma_k(\delta)$ and is defined as

Definition 5. *For $\delta \in \mathbb{C} \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+ \setminus 0$ and $\Re(\delta) > 0$, the (p,k) Gamma function ${}_p\Gamma_k(\delta)$ is given as*

$$ {}_p\Gamma_k(\delta) = \int_0^\infty e^{-\frac{t^k}{p}} t^{\delta-1} d\delta. \tag{1.9}$$

Also, he has defined the (p,k)-Pochhammer symbol ${}_p(\delta)_{n,k}$ which is given by (see [5])

Definition 6. *For $\delta \in \mathbb{C}; k, p \in \mathbb{R}^+ \setminus 0$ and $\Re(\delta) > 0, n \in \mathbb{N}$, the (p,k)- Pochhammer Symbol ${}_p(\delta)_{n,k}$ is given as*

$$\begin{aligned} {}_p(\delta)_{n,k} &= \left(\frac{\delta p}{k}\right) \cdot \left(\frac{\delta p}{k} + p\right) \cdot \left(\frac{\delta p}{k} + 2p\right) \cdots \left(\frac{\delta p}{k} + (n-1)p\right) \\ &= \frac{{}_p\Gamma_k(\delta + nk)}{{}_p\Gamma_k(\delta)}. \end{aligned} \tag{1.10}$$

The following properties hold true for the (p,k)- Gamma function and (p,k)- Pochhammer symbol given as [5]:

Proposition 2.
For $x \in \mathbb{C} \setminus k\mathbb{Z}^-; k, p \in \mathbb{R}^+$ and $\Re(x) > 0, n, q \in \mathbb{N}$, then the following formulas hold true:

$$ {}_p(\Gamma)_k(x) = \left(\frac{p}{k}\right)^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma\left(\frac{x}{k}\right). \tag{1.11}$$

$$ {}_p(x)_{nq,k} = \left(\frac{p}{k}\right)^{nq} (x)_{nq,k} = p^{nq} \left(\frac{x}{k}\right)_{nq}. \tag{1.12}$$

$$ {}_p(x)_{nq,k} = pq^{nq} \prod_{r=1}^{q} \left(\frac{\frac{x}{k} + r - 1}{q}\right)_n. \tag{1.13}$$

2. Mittag-Leffler function and its generalizations

For our present study we start by recalling the previous work. Let $\mathbb{C}, \mathbb{R}^+, \mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers and non-positive integers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Swedish mathematician G. Mittag-Leffler [6] introduced the function $E_\lambda(z)$, is defined in series form as:

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + 1)}, \qquad z \in \mathbb{C}; \lambda \geq 0. \tag{2.1}$$

The two parameter Mittag-Leffler function $E_{\lambda,\nu}(z)$ [6] studied by Wiman [7], is defined as:

$$E_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \nu)}, \qquad \lambda, \mu \in \mathbb{C};\ \Re(\lambda) > 0, \Re(\nu) > 0. \tag{2.2}$$

In [8](see also [9]), Prabhakar introduced the function $E^{\gamma}_{\lambda,\nu}(z)$, is defined as:

$$\begin{gathered} E^{\gamma}_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \nu)} \frac{z^n}{n!}, \\ \lambda, \mu, \gamma \in \mathbb{C};\ \Re(\lambda) > 0, \Re(\nu) > 0, \Re(\gamma) > 0. \end{gathered} \tag{2.3}$$

In [10], Shukla and Prajapati studied the function $E^{\gamma,\eta}_{\lambda,\nu}(z)$, is defined as (see also [11]):

$$\begin{gathered} E^{\gamma,\eta}_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\eta n}}{\Gamma(\lambda n + \nu)} \frac{z^n}{n!}, \\ \lambda, \nu, \gamma \in \mathbb{C},\ \Re(\lambda) > 0,\ \Re(\nu) > 0,\ \Re(\gamma) > 0 \ \text{ and } \ \eta \in (0,1) \cup \mathbb{N}, \end{gathered} \tag{2.4}$$

where $(\gamma)_{\eta n} = \dfrac{\Gamma(\gamma + \eta n)}{\Gamma(\gamma)}$.

Salim [12] introduced a new generalization of Mittag-Leffler function $E^{\gamma,\delta}_{\lambda,\nu}(z)$ and is defined as:

$$\begin{gathered} E^{\gamma,\delta}_{\lambda,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \nu)} \frac{z^n}{(\delta)_n}, \\ \lambda, \nu, \gamma, \delta \in \mathbb{C},\ \Re(\lambda) > 0,\ \Re(\nu) > 0,\ \Re(\gamma) > 0,\ \Re(\delta) > 0. \end{gathered} \tag{2.5}$$

Further, Salim and Faraj [13] introduced the following more generalized Mittag-Leffler function $E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(z)$ defined as:

$$\begin{gathered} E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\eta n}}{\Gamma(\lambda n + \nu)} \frac{z^n}{(\delta)_{\tau n}}, \\ \lambda, \nu, \gamma, \delta \in \mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma), \Re(\delta)\} > 0, \quad \tau, \eta \in \mathbb{R}^+, \quad \eta < \Re(\lambda) + \tau. \end{gathered} \tag{2.6}$$

Further, the following more generalized Mittag-Leffler function $E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)$ is introduced by Gupta and Parihar [14] defined as:

$$E_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\eta n,k}}{\Gamma_k(\lambda n+\nu)} \frac{z^n}{(\delta)_{\tau n,k}}, \tag{2.7}$$
$$\lambda,\nu,\gamma,\delta \in \mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma), \Re(\delta)\} > 0, \quad k,\tau,\eta \in \mathbb{R}^+, \quad \eta < \Re(\lambda)+\tau.$$

Here we introduce a new generalization of Mittag-Leffler function defined as:

$${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)} \frac{z^n}{{}_p(\delta)_{\tau n,k}}, \tag{2.8}$$
$$\lambda,\nu,\gamma,\delta \in \mathbb{C},\ \min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma), \Re(\delta)\} > 0, \quad p,k,\tau,\eta \in \mathbb{R}^+, \quad \eta < \frac{\Re(\lambda)}{k}+\tau.$$

where ${}_p(\gamma)_{\eta n,k}$ is the (p,k)- Pochhammer symbol defined in equation (1.10).

Also we require an interesting generalization of the generalized hypergeometric series ${}_pF_q$ (see e.g., [15]) due to Fox [16] and Wright [17, 18, 19] known as the Fox-Wright function ${}_p\Psi_q(z)$ $(p,q \in \mathbb{N}_0)$ with p numerator and q denominator parameters defined for $a_1,...,a_p \in \mathbb{C}$ and $b_1,...,b_q \in \mathbb{C}\backslash\mathbb{Z}_0^-$ by (For details see [3, 20, 21, 22])

$${}_p\Psi_q\left[\begin{array}{c}(a_1,\alpha_1),...,(a_p,\alpha_p);\\(b_1,\beta_1),...,(b_q,\beta_q);\end{array} z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 n)...\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)...\Gamma(b_q+\beta_q n)} \frac{z^n}{n!}, \tag{2.9}$$

where the coefficients $\alpha_1,...,\alpha_p,\ \beta_1,...,\beta_q \in \mathbb{R}^+$ are such that

$$1+\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i \geq 0. \tag{2.10}$$

For $\alpha_i = \beta_j = 1$ $(i=1,...,p; j=1,...,q)$, equation (2.9) reduces immediately to the generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N}_0)$ (see [22]):

$${}_pF_q\left[\begin{array}{c}a_1,...,a_p;\\b_1,...,b_q;\end{array} z\right] = \frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_q)}{}_p\Psi_q\left[\begin{array}{c}(a_1,1),...,(a_p,1);\\(b_1,1),...,(b_q,1);\end{array} z\right]. \tag{2.11}$$

3. Basic properties of the extended generalized Mittag-Leffler function

In this section we investigate several interesting properties of the extended generalized Mittag-Leffler function ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$.

Theorem 1. *The series in equation* (2.8) *is absolutely convergent for all values of* z *provided* $\eta < \frac{\Re(\lambda)}{k}+\tau$. *Moreover if* $\eta = \frac{\Re(\lambda)}{k}+\tau$, *then* ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ *converges for* $|z|<1$.

Proof. Rewriting ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ in the form of power series ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z) = \sum_{n=0}^{\infty} b_n z^n$,

where $b_n = \frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu){}_p(\delta)_{\tau n,k}}$.

Applying

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2z}+\mathsf{O}\left(\frac{1}{z^2}\right)\right], \tag{3.1}$$

we get

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{{}_p(\gamma)_{\eta n+\eta,k}}{{}_p(\gamma)_{\eta n,k}}\frac{{}_p(\delta)_{\tau n,k}}{{}_p(\delta)_{\tau n+\tau,k}}\frac{{}_p\Gamma_k(\lambda n+\nu)}{{}_p\Gamma_k(\lambda n+\lambda+\nu)}\frac{z^{n+1}}{z^n}\right| \\
&= p^{\eta-\tau-\frac{\lambda}{k}}\,(\eta n)^{\eta}\left[1+\frac{\eta\left(\frac{2\gamma}{k}+\eta-1\right)}{2\eta n}+\mathsf{O}\left(\frac{1}{(\eta n)^2}\right)\right] \\
&\quad\times\left(\frac{\lambda n}{k}\right)^{-\frac{\lambda}{k}}\left[1+\frac{-\frac{\lambda}{k}\left(\frac{2\nu}{k}+\frac{\lambda}{k}-1\right)}{\frac{2\lambda n}{k}}+\mathsf{O}\left(\frac{1}{\left(\frac{\lambda n}{k}\right)^2}\right)\right] \\
&\quad\times(\tau n)^{-\tau}\left[1+\frac{-\tau\left(\frac{2\delta}{k}+\tau-1\right)}{2\tau n}+\mathsf{O}\left(\frac{1}{(\tau n)^2}\right)\right]|z| \\
&= p^{\eta-\tau-\frac{\lambda}{k}}\frac{\eta^{\eta}}{\frac{\lambda}{k}^{\frac{\lambda}{k}}\tau^{\tau}}n^{\frac{\eta}{\frac{\lambda}{k}+\tau}}\left[1+\frac{\eta\left(\frac{2\gamma}{k}+\eta-1\right)}{2\eta n}+\mathsf{O}\left(\frac{1}{(\eta n)^2}\right)\right] \\
&\quad\times\left[1+\frac{-\frac{\lambda}{k}\left(\frac{2\nu}{k}+\frac{\lambda}{k}-1\right)}{\frac{2\lambda n}{k}}+\mathsf{O}\left(\frac{1}{\left(\frac{\lambda n}{k}\right)^2}\right)\right] \\
&\quad\times\left[1+\frac{-\tau\left(\frac{2\delta}{k}+\tau-1\right)}{2\tau n}+\mathsf{O}\left(\frac{1}{(\tau n)^2}\right)\right]|z|.
\end{aligned} \tag{3.2}$$

Here $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow 0$ as $n \longrightarrow \infty$ when $\eta < \frac{\Re(\lambda)}{k}+\tau$.

that implies the function ${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)$ converges for all z provided $\eta < \frac{\Re(\lambda)}{k}+\tau$. Moreover if $\eta = \frac{\Re(\lambda)}{k}+\tau$, then ${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)$ converges for $|z|<1$. □

Theorem 2. *Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda),\Re(\nu),\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k}+\tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$$\left(\frac{d}{dz}\right)^m {}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \Gamma(m+1)\frac{{}_p(\gamma)_{\eta m,k}}{{}_p(\delta)_{\tau m,k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma+\eta mk)_{\eta n,k}(m+1)_n}{{}_p\Gamma_k(\lambda n+\lambda m+\nu)_p(\delta+\tau mk)_{\tau n,k}}\frac{z^n}{n!}. \tag{3.3}$$

$$\left(\frac{d}{dz}\right)^m z^{\frac{\nu}{k}}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(\omega z^{\frac{\lambda}{k}}) = \left(\frac{\omega}{p}\right)^m z^{\frac{(\lambda-k)m+\nu}{k}}\frac{{}_p(\gamma)_{\eta m,k}}{{}_p(\delta)_{\tau m,k}}{}_pE^{\gamma+\eta mk,\delta+\tau mk,\eta}_{k,\lambda,\nu+k(1-m)+\lambda m,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \tag{3.4}$$

Proof. From equation (3.3), using equation (1.10), (1.11) and (1.12), we have

$$\left(\frac{d}{dz}\right)^m {}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \left(\frac{d}{dz}\right)^m (z)^n$$

$$= \sum_{n=m}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \frac{\Gamma(n+1)}{\Gamma(n-m+1)} (z)^{n-m}$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta(n+m),k}}{{}_p\Gamma_k(\lambda(n+m)+\nu)_p(\delta)_{\tau(n+m),k}} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} (z)^{n}$$

$$= \Gamma(m+1)\frac{{}_p(\gamma)_{\eta m,k}}{{}_p(\delta)_{\tau m,k}} \sum_{n=0}^{\infty} \frac{{}_p(\gamma+\eta mk)_{\eta n,k}(m+1)_n}{{}_p\Gamma_k(\lambda n+\lambda m+\nu)_p(\delta+\tau mk)_{\tau n,k}} \frac{z^n}{n!}. \tag{3.5}$$

From equation (3.4), using equation (1.10), (1.11) and (1.12), we have

$$\left(\frac{d}{dz}\right)^m z^{\frac{\nu}{k}}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(\omega z^{\frac{\lambda}{k}}) = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}(\omega)^n z^{\frac{\lambda n+\nu}{k}}}{{}_p\Gamma_k(\lambda n+\nu+k)_p(\delta)_{\tau n,k}}$$

$$= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}(\omega)^n}{{}_p\Gamma_k(\lambda n+\nu+k)_p(\delta)_{\tau n,k}} \left(\frac{d}{dz}\right)^m (z)^{\frac{\lambda n+\nu}{k}}$$

$$= \sum_{n=m}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}(\omega)^n}{\frac{p^{\frac{\lambda n+\nu+k}{k}}}{k}\Gamma\left(\frac{\lambda n+\nu+k}{k}\right){}_p(\delta)_{\tau n,k}} \frac{\Gamma(\frac{\lambda n+\nu}{k}+1)}{\Gamma(\frac{\lambda n+\nu}{k}+1-m)} (z)^{\frac{\lambda n+\nu}{k}-m}$$

$$= \sum_{n=0}^{\infty} \frac{p^{-m}{}_p(\gamma)_{\eta(n+m),k}(\omega)^{n+m}(z)^{\frac{\lambda(n+m)+\nu}{k}-m}}{{}_p\Gamma_k(\lambda(n+m)+\nu+k-km)_p(\delta)_{\tau(n+m),k}}$$

$$= \left(\frac{\omega}{p}\right)^m z^{\frac{(\lambda-k)m+\nu}{k}} \frac{{}_p(\gamma)_{\eta m,k}}{{}_p(\delta)_{\tau m,k}} \sum_{n=0}^{\infty} \frac{{}_p(\gamma+\eta mk)_{\eta n,k}\left(\omega z^{\frac{\lambda}{k}}\right)^n}{{}_p\Gamma_k(\lambda n+\lambda m+\nu+k-km)_p(\delta+\tau mk)_{\tau n,k}}$$

$$= \left(\frac{\omega}{p}\right)^m z^{\frac{(\lambda-k)m+\nu}{k}} \frac{{}_p(\gamma)_{\eta m,k}}{{}_p(\delta)_{\tau m,k}} {}_pE^{\gamma+\eta mk,\delta+\tau mk,\eta}_{k,\lambda,\nu+k(1-m)+\lambda m,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \tag{3.6}$$

□

Theorem 3. *Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $p,k,\tau,\eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k}+\tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{p\nu}{k}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z) + \frac{p\lambda z}{k}\frac{d}{dz}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z). \tag{3.7}$$

$${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - {}_pE^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{z\eta k}{\gamma-k}\frac{d}{dz}{}_pE^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z). \tag{3.8}$$

$${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - {}_pE^{\gamma,\delta-k,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{z\tau k}{k-\delta}\frac{d}{dz}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z). \tag{3.9}$$

Proof. From equation (3.7), using equation (1.10), (1.11) and (1.12), we have

$$
\begin{aligned}
{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) &= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k} \left(\frac{\lambda n+\nu}{k}\right)\, z^n}{\frac{p^{\frac{\lambda n+\nu}{k}}}{k}\Gamma\left(\frac{\lambda n+\nu}{k}\right)\left(\frac{\lambda n+\nu}{k}\right){}_p(\delta)_{\tau n,k}} \\
&= \frac{p\nu}{k}\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu+k)_p(\delta)_{\tau n,k}} \\
&\quad - \frac{p\lambda z}{k}\frac{d}{dz}\sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu+k)_p(\delta)_{\tau n,k}} \\
&= \frac{p\nu}{k}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z) + \frac{p\lambda z}{k}\frac{d}{dz}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z).
\end{aligned} \tag{3.10}
$$

And from equation (3.8), using equation (1.10), (1.11) and (1.12), we have

$$
\begin{aligned}
{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - {}_pE^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z) &= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} - \sum_{n=0}^{\infty} \frac{{}_p(\gamma-k)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \\
&= \sum_{n=0}^{\infty} \frac{z^n[{}_p(\gamma)_{\eta n,k} - {}_p(\gamma-k)_{\eta n,k}]}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma-k)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\left[\frac{\eta n k}{\gamma-k}\right] \\
&= \frac{\eta k z}{\gamma-k}\sum_{n=1}^{\infty} \frac{{}_p(\gamma-k)_{\eta n,k} n\, z^{n-1}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} \\
&= \frac{z\eta k}{\gamma-k}\frac{d}{dz}{}_pE^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z).
\end{aligned} \tag{3.11}
$$

Also from equation (3.9), using equation (1.10), (1.11) and (1.12), we have

$$
\begin{aligned}
&{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - {}_pE^{\gamma,\delta-k,\eta}_{k,\lambda,\nu,\tau}(z) \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}} - \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta-k)_{\tau n,k}} \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)}\left[\frac{1}{{}_p(\delta)_{\tau n,k}} - \frac{1}{{}_p(\delta-k)_{\tau n,k}}\right] \\
&= \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\left[\frac{\tau n k}{k-\delta}\right] \\
&= \frac{z\tau k}{k-\delta}\sum_{n=1}^{\infty} \frac{{}_p(\gamma)_{\eta n,k} n\, z^{n-1}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}
\end{aligned}
$$

$$= \frac{z\tau k}{k-\delta}\frac{d}{dz}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z). \tag{3.12}$$

□

Theorem 4. *Let* $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, *then, the following formula holds true:*

$$\frac{1}{\Gamma\left(\frac{\alpha}{k}\right)}\int_0^1 u^{\frac{\nu}{k}-1}(1-u)^{\frac{\alpha}{k}-1}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(zu^{\frac{\lambda}{k}}\right)du = p^{\frac{\alpha}{k}}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}(z). \tag{3.13}$$

Proof. Denote left hand side of equation (3.13) by I, using equation (1.10), (1.11) and (1.12), we have

$$\begin{aligned} I &= \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^1 u^{\frac{\lambda n}{k}+\frac{\nu}{k}-1}(1-u)^{\frac{\alpha}{k}-1}du \\ &= \frac{1}{\Gamma(\frac{\alpha}{k})}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\frac{\Gamma\left(\frac{\lambda n}{k}+\frac{\nu}{k}\right)\Gamma\left(\frac{\alpha}{k}\right)}{\Gamma\left(\frac{\lambda n}{k}+\frac{\nu}{k}+\frac{\alpha}{k}\right)} \\ &= p^{\frac{\alpha}{k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,z^n}{{}_p\Gamma_k\left(\frac{\lambda n}{k}+\frac{\nu}{k}+\frac{\alpha}{k}\right){}_p(\delta)_{\tau n,k}} \\ &= p^{\frac{\alpha}{k}}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}(z). \end{aligned} \tag{3.14}$$

□

Theorem 5. *Let* $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\,\Re(\nu),\,\Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, *then, the following formula holds true:*

$$\frac{1}{\Gamma\left(\frac{\alpha}{k}\right)}\int_t^x(x-s)^{\frac{\alpha}{k}-1}(s-t)^{\frac{\nu}{k}-1}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\xi(s-t)^{\frac{\lambda}{k}}\right)ds = p^{\frac{\alpha}{k}}(x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}\left(\xi(x-t)^{\frac{\lambda}{k}}\right). \tag{3.15}$$

Proof. Denote left hand side of equation (3.15) by I and putting $u=\dfrac{s-t}{x-t}$, using equation (1.10), (1.11) and (1.12), we have

$$\begin{aligned} I &= \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)}\int_0^1(x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1}u^{\frac{\lambda n}{k}+\frac{\nu}{k}-1}(1-u)^{\frac{\alpha}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\left(\xi(x-t)^{\frac{\lambda}{k}}\right)^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}du \\ &= \frac{(x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1}}{\Gamma\left(\frac{\alpha}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\left(\xi(x-t)^{\frac{\lambda}{k}}\right)^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^1 u^{\frac{\lambda n}{k}+\frac{\nu}{k}-1}(1-u)^{\frac{\alpha}{k}-1}du \\ &= \frac{(x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1}}{\Gamma\left(\frac{\alpha}{k}\right)}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\left(\xi(x-t)^{\frac{\lambda}{k}}\right)^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\frac{\Gamma\left(\frac{\lambda n}{k}+\frac{\nu}{k}\right)\Gamma\left(\frac{\alpha}{k}\right)}{\Gamma\left(\frac{\lambda n}{k}+\frac{\nu}{k}+\frac{\alpha}{k}\right)} \\ &= p^{\frac{\alpha}{k}}(x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}\left(\xi(x-t)^{\frac{\lambda}{k}}\right). \end{aligned} \tag{3.16}$$

□

Theorem 6. *Let $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^{+}$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then, the following formula holds true:*

$$\int_0^z t^{\frac{\nu}{k}-1}\,{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\omega t^{\frac{\lambda}{k}}\right)dt = pz^{\frac{\nu}{k}}\,{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \tag{3.17}$$

Proof. Denote left hand side of equation (3.17) by I, using equation (1.10), (1.11) and (1.12), we have

$$\begin{aligned} I &= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,\omega^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^z t^{\frac{\lambda n}{k}+\frac{\nu}{k}-1}dt\\ &= z^{\frac{\nu}{k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\frac{\left(\omega z^{\frac{\lambda}{k}}\right)^n}{\left(\frac{\lambda n}{k}+\frac{\nu}{k}\right)}\\ &= pz^{\frac{\nu}{k}}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\left(\omega z^{\frac{\lambda}{k}}\right)^n}{{}_p\Gamma_k\left(\lambda n+\nu+k\right){}_p(\delta)_{\tau n,k}}\\ &= pz^{\frac{\nu}{k}}\,{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \end{aligned} \tag{3.18}$$

□

4. Generalized hypergeometric function representation of the extended generalized Mittag-Leffler function

Theorem 7. *Let $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^{+}$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then, the following formula holds true:*

$$ {}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)=\frac{1}{{}_p\Gamma_k(\nu)}{}_{\eta+1}F_{\frac{\lambda}{k}+\tau}\left[\begin{array}{ll}\Delta\left(\eta;\frac{\gamma}{k}\right), & 1;\\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right),\Delta\left(\tau;\frac{\delta}{k}\right); & \end{array} z\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}k^{\frac{\lambda}{k}}}{\tau^{\tau}\lambda^{\frac{\lambda}{k}}}\right]. \tag{4.1}$$

where $\Delta\left(m;l\right)$ *is* m*-tuple* $\frac{l}{m},\frac{l+1}{m},\frac{l+2}{m},...,\frac{l+m-1}{m}$.

Proof. From equation (4.1), using equation (1.10), (1.11), (1.12) and (1.13), we have

$$ {}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z)=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}$$

$$=\frac{1}{{}_p\Gamma_k(\nu)}\sum_{n=0}^{\infty}\frac{(p\eta)^{\eta n}\prod_{i=1}^{\eta}\left(\frac{\frac{\gamma}{k}+i-1}{\eta}\right)_n(1)_n}{\left(\frac{p\lambda}{k}\right)^{\frac{\lambda n}{k}}\prod_{j=1}^{\frac{\lambda}{k}}\left(\frac{\frac{\nu}{k}+j-1}{\frac{\lambda}{k}}\right)_n(p\tau)^{\tau n}\prod_{r=1}^{\tau}\left(\frac{\frac{\delta}{k}+r-1}{\tau}\right)_n}\frac{z^n}{n!}$$

$$= \frac{1}{{}_p\Gamma_k(\nu)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{\eta} \left(\frac{\frac{\gamma}{k}+i-1}{\eta}\right)_n (1)_n}{\prod_{j=1}^{\frac{\lambda}{k}} \left(\frac{\frac{\nu}{k}+j-1}{\frac{\lambda}{k}}\right)_n \prod_{r=1}^{\tau} \left(\frac{\frac{\delta}{k}+r-1}{\tau}\right)_n} \left(z\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}k^{\frac{\lambda}{k}}}{\tau^{\tau}\lambda^{\frac{\lambda}{k}}}\right)^n \frac{1}{n!}$$

$$= \frac{1}{{}_p\Gamma_k(\nu)} {}_{\eta+1}F_{\frac{\lambda}{k}+\tau} \left[\begin{array}{ll} \Delta\left(\eta;\frac{\gamma}{k}\right), & 1; \\ \Delta\left(\frac{\lambda}{k};\frac{\nu}{k}\right), \Delta\left(\tau;\frac{\delta}{k}\right); \end{array} z\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}k^{\frac{\lambda}{k}}}{\tau^{\tau}\lambda^{\frac{\lambda}{k}}} \right]. \tag{4.2}$$

□

5. Extended generalized Mittag-Leffler function in terms of generalized Fox-Wright function

Theorem 8. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $p, k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, the following formula holds true:*

$${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{k\Gamma\left(\frac{\delta}{k}\right)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{array}{l} \left(\frac{\gamma}{k},\eta\right), (1,1); \\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), \left(\frac{\delta}{k},\tau\right); \end{array} zp^{\eta-\tau-\frac{\lambda}{k}} \right]. \tag{5.1}$$

Proof. From equation (5.1), using equation (1.10), (1.11) and (1.12), we have

$${}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{\eta n,k}\, z^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}$$

$$= \frac{k\Gamma\left(\frac{\delta}{k}\right)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+\eta n\right)\Gamma(1+n)}{\Gamma\left(\frac{\lambda n}{k}+\frac{\nu}{k}\right)\Gamma\left(\frac{\delta}{k}+\tau n\right)} \frac{(zp^{\eta-\tau-\frac{\lambda}{k}})^n}{n!}$$

$$= \frac{k\Gamma\left(\frac{\delta}{k}\right)}{p^{\frac{\nu}{k}}\Gamma\left(\frac{\gamma}{k}\right)} {}_2\Psi_2 \left[\begin{array}{l} \left(\frac{\gamma}{k},\eta\right), (1,1); \\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), \left(\frac{\delta}{k},\tau\right); \end{array} zp^{\eta-\tau-\frac{\lambda}{k}} \right].$$

□

5.1. *Special Cases*

On setting $p = 1$, results presented in Theorem 1–Theorem 6 reduce to the following form.

Corollary 1. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$$\left(\frac{d}{dz}\right)^m E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \Gamma(m+1)\frac{(\gamma)_{\eta m,k}}{(\delta)_{\tau m,k}} \sum_{n=0}^{\infty} \frac{(\gamma+\eta mk)_{\eta n,k}(m+1)_n}{\Gamma_k(\lambda n+\lambda m+\nu)(\delta+\tau mk)_{\tau n,k}} \frac{z^n}{n!}. \tag{5.2}$$

$$\left(\frac{d}{dz}\right)^m z^{\frac{\nu}{k}} E^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(\omega z^{\frac{\lambda}{k}}) = \omega^m z^{\frac{(\lambda-k)m+\nu}{k}} \frac{(\gamma)_{\eta m,k}}{(\delta)_{\tau m,k}} E^{\gamma+\eta mk,\delta+\tau mk,\eta}_{k,\lambda,\nu+k(1-m)+\lambda m,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \tag{5.3}$$

Corollary 2. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$$E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{\nu}{k} E^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z) + \frac{\lambda z}{k} \frac{d}{dz} E^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}(z). \tag{5.4}$$

$$E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - E^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{z\eta k}{\gamma - k} \frac{d}{dz} E^{\gamma-k,\delta,\eta}_{k,\lambda,\nu,\tau}(z). \tag{5.5}$$

$$E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z) - E^{\gamma,\delta-k,\eta}_{k,\lambda,\nu,\tau}(z) = \frac{z\tau k}{k - \delta} \frac{d}{dz} E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}(z). \tag{5.6}$$

Corollary 3. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, the following formula holds true:*

$$\frac{1}{\Gamma\left(\frac{\alpha}{k}\right)} \int_0^1 u^{\frac{\nu}{k}-1}(1-u)^{\frac{\alpha}{k}-1} E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(zu^{\frac{\lambda}{k}}\right) du = E^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}(z). \tag{5.7}$$

Corollary 4. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, the following formula holds true:*

$$\begin{aligned} \frac{1}{\Gamma\left(\frac{\alpha}{k}\right)} \int_t^x (x-s)^{\frac{\alpha}{k}-1}(s-t)^{\frac{\nu}{k}-1} E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\xi(s-t)^{\frac{\lambda}{k}}\right) ds \\ = (x-t)^{\frac{\alpha}{k}+\frac{\nu}{k}-1} E^{\gamma,\delta,\eta}_{k,\lambda,\nu+\alpha,\tau}\left(\xi(x-t)^{\frac{\lambda}{k}}\right). \end{aligned} \tag{5.8}$$

Corollary 5. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, the following formula holds true:*

$$\int_0^z t^{\frac{\nu}{k}-1} E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(\omega t^{\frac{\lambda}{k}}\right) dt = z^{\frac{\nu}{k}} E^{\gamma,\delta,\eta}_{k,\lambda,\nu+k,\tau}\left(\omega z^{\frac{\lambda}{k}}\right). \tag{5.9}$$

On setting $p = k = 1$, results presented in Theorem 1– Theorem 6 reduce to the following form.

Corollary 6. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau, \eta \in \mathbb{R}^+$, $\eta < \Re(\lambda) + \tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$$\left(\frac{d}{dz}\right)^m E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(z) = \Gamma(m+1) \frac{(\gamma)_{\eta m}}{(\delta)_{\tau m}} \sum_{n=0}^{\infty} \frac{(\gamma+\eta m)_{\eta n}(m+1)_n}{\Gamma(\lambda n + \lambda m + \nu)(\delta + \tau m)_{\tau n}} \frac{z^n}{n!}. \tag{5.10}$$

$$\left(\frac{d}{dz}\right)^m z^\nu E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(\omega z^\lambda) = \omega^m z^{(\lambda-1)m+\nu} \frac{(\gamma)_{\eta m}}{(\delta)_{\tau m}} E^{\gamma+\eta m,\delta+\tau m,\eta}_{\lambda,\nu+1-m+\lambda m,\tau}\left(\omega z^\lambda\right). \tag{5.11}$$

Corollary 7. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau, \eta \in \mathbb{R}^+$, $\eta < \Re(\lambda) + \tau$, *then, for* $m \in \mathbb{N}$, *the following formula holds true:*

$$E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(z) = \nu E^{\gamma,\delta,\eta}_{\lambda,\nu+1,\tau}(z) + \lambda z \frac{d}{dz} E^{\gamma,\delta,\eta}_{\lambda,\nu+1,\tau}(z). \tag{5.12}$$

$$E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}(z) - E^{\gamma-1,\delta,\eta}_{\lambda,\nu,\tau}(z) = \frac{z\eta}{\gamma - 1} \frac{d}{dz} E^{\gamma-1,\delta,\eta}_{\lambda,\nu,\tau}(z). \tag{5.13}$$

$$E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z) - E_{\lambda,\nu,\tau}^{\gamma,\delta-1,\eta}(z) = \frac{z\tau}{1-\delta}\frac{d}{dz}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z). \tag{5.14}$$

Corollary 8. *Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau,\eta \in \mathbb{R}^+$, $\eta < \Re(\lambda)+\tau$, *then, the following formula holds true:*

$$\frac{1}{\Gamma(\alpha)}\int_0^1 u^{\nu-1}(1-u)^{\alpha-1}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(zu^{\lambda}\right)du = E_{\lambda,\nu+\alpha,\tau}^{\gamma,\delta,\eta}(z). \tag{5.15}$$

Corollary 9. *Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau,\eta \in \mathbb{R}^+$, $\eta < \Re(\lambda)+\tau$, *then, the following formula holds true:*

$$\frac{1}{\Gamma(\alpha)}\int_t^x (x-s)^{\alpha-1}(s-t)^{\nu-1}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\xi(s-t)^{\lambda}\right)ds = (x-t)^{\alpha+\nu-1}E_{k,\lambda,\nu+\alpha,\tau}^{\gamma,\delta,\eta}\left(\xi(x-t)^{\lambda}\right). \tag{5.16}$$

Corollary 10. *Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau,\eta \in \mathbb{R}^+$, $\eta < \Re(\lambda)+\tau$, *then, the following formula holds true:*

$$\int_0^z t^{\nu-1}E_{\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(\omega t^{\lambda}\right)dt = z^{\nu}E_{\lambda,\nu+1,\tau}^{\gamma,\delta,\eta}\left(\omega z^{\lambda}\right). \tag{5.17}$$

6. Integral transforms of the extended generalized Mittag-Leffler function

In this section, we present certain image formulas of the extended generalized Mittag-Leffler function ${}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}(z)$ by using integral transforms.

Theorem 9. *(***Laplace transform***) Let* $\lambda,\nu,\gamma,\delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $p,k,\tau,\eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k}+\tau$, *then, for* $\Re(s) > 0$, *the following formula holds true:*

$$L\left\{z^{\frac{\nu}{k}-1}{}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(xz^{\frac{\lambda}{k}}\right);s\right\} = \frac{k}{(sp)^{\frac{\nu}{k}}}\,{}_{\eta+1}F_{\tau}\left[\begin{array}{l}\Delta\left(\eta;\frac{\gamma}{k}\right),1;\\ \Delta\left(\tau;\frac{\delta}{k}\right);\end{array}\; x\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}}{\tau^{\tau}s^{\frac{\lambda}{k}}}\right]. \tag{6.1}$$

Proof.

$$L\left\{z^{\frac{\nu}{k}-1}{}_pE_{k,\lambda,\nu,\tau}^{\gamma,\delta,\eta}\left(xz^{\frac{\lambda}{k}}\right);s\right\} = \int_0^{\infty} e^{-sz}z^{\frac{\nu}{k}-1}\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\left(xz^{\frac{\lambda}{k}}\right)^n dz$$

$$= \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,x^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^{\infty} e^{-sz}z^{\frac{\lambda n+\nu}{k}-1}dz$$

$$= \frac{k}{(sp)^{\frac{\nu}{k}}}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{\eta}\left(\frac{\frac{\gamma}{k}+i-1}{\eta}\right)_n (1)_n}{\prod_{j=1}^{\tau}\left(\frac{\frac{\delta}{k}+j-1}{\tau}\right)_n}\left(x\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}}{\tau^{\tau}s^{\frac{\lambda}{k}}}\right)^n\frac{1}{n!}$$

$$= \frac{k}{(sp)^{\frac{\nu}{k}}}\ {}_{\eta+1}F_{\tau}\left[\begin{array}{c}\Delta\left(\eta;\frac{\gamma}{k}\right),1;\\ \Delta\left(\tau;\frac{\delta}{k}\right);\end{array}\ x\frac{p^{\eta-\tau-\frac{\lambda}{k}}\eta^{\eta}}{\tau^{\tau}s^{\frac{\lambda}{k}}}\right]. \tag{6.2}$$

□

Theorem 10. *(***Beta transform***) Let $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then, for $\Re(a)>0,\Re(b)>0$, the following formula holds true:*

$$B\left\{{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\frac{\lambda}{k}}\right);\frac{\nu}{k},\frac{\vartheta}{k}\right\}=k_p\Gamma_k(\vartheta)_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\vartheta,\tau}(x). \tag{6.3}$$

Proof.

$$\begin{aligned}B\left\{{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\frac{\lambda}{k}}\right);\frac{\nu}{k},\frac{\vartheta}{k}\right\}&=\int_0^1 z^{\frac{\nu}{k}-1}(1-z)^{\frac{\vartheta}{k}-1}{}_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\frac{\lambda}{k}}\right)dz\\
&=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,x^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^1 z^{\frac{\lambda n+\nu}{k}-1}(1-z)^{\frac{\vartheta}{k}-1}dz\\
&=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,x^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\frac{\Gamma\left(\frac{\lambda n+\nu}{k}\right)\Gamma\left(\frac{\vartheta}{k}\right)}{\Gamma\left(\frac{\lambda n+\nu+\vartheta}{k}\right)}\\
&=k_p\Gamma_k(\vartheta)\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,x^n}{{}_p\Gamma_k(\lambda n+\nu+\vartheta)_p(\delta)_{\tau n,k}}\\
&=k_p\Gamma_k(\vartheta)_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu+\vartheta,\tau}(x). \qquad (6.4)\end{aligned}$$

□

Theorem 11. *(***Whittaker Transform***) Let $\lambda,\nu,\gamma,\delta\in\mathbb{C}$, $\min\{\Re(\lambda),\ \Re(\nu),\ \Re(\gamma),\Re(\delta)\}>0$, $p,k,\tau,\eta\in\mathbb{R}^+$, $\eta<\frac{\Re(\lambda)}{k}+\tau$, then, for $\Re(a)>0,\Re(b)>0$, the following formula holds true:*

$$\begin{aligned}&\int_0^{\infty}z^{l-1}e^{-\zeta z/2}W_{a,b}(\zeta z)_pE^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\vartheta}\right)dz\\
&=\frac{\Gamma\left(\frac{\delta}{k}\right)k\zeta^{-l}}{\Gamma\left(\frac{\gamma}{k}\right)p^{\frac{\nu}{k}}}{}_4\Psi_3\left[\begin{array}{c}\left(\frac{\gamma}{k},\eta\right),\left(\frac12+b+l,\vartheta\right),\left(\frac12-b+l,\vartheta\right),(1,1);\\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right),\left(\frac{\delta}{k},\tau\right),\left(\frac12-a+l,\vartheta\right);\end{array}\left(\frac{xp^{\eta-\tau-\frac{\lambda}{k}}}{\zeta^{\vartheta}}\right)\right].\end{aligned} \tag{6.5}$$

Proof. Denote left hand side of equation (6.5) by I, we have

$$I=\sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{\eta n,k}\,x^n}{{}_p\Gamma_k(\lambda n+\nu)_p(\delta)_{\tau n,k}}\int_0^{\infty}z^{l+\vartheta n-1}e^{-\zeta z/2}W_{a,b}(\zeta z)dz, \tag{6.6}$$

on setting $\zeta z=y$ and using equation (1.10), (1.11) and (1.12), we get

$$\begin{aligned}I=&\frac{\Gamma\left(\frac{\delta}{k}\right)k\zeta^{-l}}{\Gamma\left(\frac{\gamma}{k}\right)p^{\frac{\nu}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma}{k}+\eta n\right)\Gamma(1+n)}{\Gamma\left(\frac{\nu}{k}+\frac{\lambda}{k}n\right)\Gamma\left(\frac{\delta}{k}+\tau n\right)}\left(\frac{xp^{\eta-\tau-\frac{\lambda}{k}}}{\zeta^{\vartheta}}\right)^n\frac{1}{n!}\\
&\times\int_0^{\infty}y^{l+\vartheta n-1}e^{-y/2}W_{a,b}(y)dy,\end{aligned} \tag{6.7}$$

after simplification, we get

$$I = \frac{\Gamma\left(\frac{\delta}{k}\right) k\zeta^{-l}}{\Gamma\left(\frac{\gamma}{k}\right) p^{\frac{\nu}{k}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+\eta n\right)\Gamma\left(\frac{1}{2}+b+l+\vartheta n\right)}{\Gamma\left(\frac{\nu}{k}+\frac{\lambda}{k}n\right)\Gamma\left(\frac{\delta}{k}+\tau n\right)} \times \frac{\Gamma\left(\frac{1}{2}-b+l+\vartheta n\right)\Gamma(1+n)}{\Gamma\left(\frac{1}{2}-a+l+\vartheta n\right)} \left(\frac{xp^{\eta-\tau-\frac{\lambda}{k}}}{\zeta^{\vartheta}}\right)^{n} \frac{1}{n!}, \tag{6.8}$$

using equation (2.9), we get the required result (6.5)

$$I = \frac{\Gamma\left(\frac{\delta}{k}\right) k\zeta^{-l}}{\Gamma\left(\frac{\gamma}{k}\right) p^{\frac{\nu}{k}}} {}_4\Psi_3 \left[\begin{array}{c} \left(\frac{\gamma}{k},\eta\right), \left(\frac{1}{2}+b+l,\vartheta\right), \left(\frac{1}{2}-b+l,\vartheta\right), (1,1); \\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), \left(\frac{\delta}{k},\tau\right), \left(\frac{1}{2}-a+l,\vartheta\right); \end{array} \left(\frac{xp^{\eta-\tau-\frac{\lambda}{k}}}{\zeta^{\vartheta}}\right) \right]. \tag{6.9}$$

□

6.1. Special Cases

Here we present some special cases by choosing suitable values of the parameters involved.

On setting $p = 1$ results presented in Theorem 9, Theorem 10 and Theorem 11 reduce to the following form.

Corollary 11. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, for* $\Re(s) > 0$, *the following formula holds true:*

$$L\left\{z^{\frac{\nu}{k}-1} E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\frac{\lambda}{k}}\right); s\right\} = \frac{k}{(s)^{\frac{\nu}{k}}} {}_{\eta+1}F_{\tau} \left[\begin{array}{c} \Delta\left(\eta; \frac{\gamma}{k}\right), 1; \\ \Delta\left(\tau; \frac{\delta}{k}\right); \end{array} \frac{x\eta^{\eta}}{\tau^{\tau} s^{\frac{\lambda}{k}}} \right]. \tag{6.10}$$

Corollary 12. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, for* $\Re(a) > 0, \Re(b) > 0$, *the following formula holds true:*

$$B\left\{E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\frac{\lambda}{k}}\right); \frac{\nu}{k}, \frac{\vartheta}{k}\right\} = k\Gamma_k(\vartheta) E^{\gamma,\delta,\eta}_{k,\lambda,\nu+\vartheta,\tau}(x). \tag{6.11}$$

Corollary 13. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $k, \tau, \eta \in \mathbb{R}^+$, $\eta < \frac{\Re(\lambda)}{k} + \tau$, *then, for* $\Re(a) > 0, \Re(b) > 0$, *the following formula holds true:*

$$\int_0^{\infty} z^{l-1} e^{-\zeta z/2} W_{a,b}(\zeta z) E^{\gamma,\delta,\eta}_{k,\lambda,\nu,\tau}\left(xz^{\vartheta}\right) dz = \frac{\Gamma\left(\frac{\delta}{k}\right) k\zeta^{-l}}{\Gamma\left(\frac{\gamma}{k}\right)} {}_4\Psi_3 \left[\begin{array}{c} \left(\frac{\gamma}{k},\eta\right), \left(\frac{1}{2}+b+l,\vartheta\right), \left(\frac{1}{2}-b+l,\vartheta\right), (1,1); \\ \left(\frac{\nu}{k},\frac{\lambda}{k}\right), \left(\frac{\delta}{k},\tau\right), \left(\frac{1}{2}-a+l,\vartheta\right); \end{array} \left(\frac{x}{\zeta^{\vartheta}}\right) \right]. \tag{6.12}$$

On setting $p = k = 1$, results presented in Theorem 9, Theorem 10 and Theorem 11 reduce to the following form.

Corollary 14. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau, \eta \in \mathbb{R}^+$, $\eta < \Re(\lambda) + \tau$, *then, for* $\Re(s) > 0$, *the following formula holds true:*

$$L\left\{z^{\nu-1}E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}\left(xz^{\lambda}\right); s\right\} = \frac{1}{s^{\nu}}\,{}_{\eta+1}F_{\tau}\left[\begin{array}{c}\Delta(\eta;\gamma), 1;\\ \Delta(\tau;\delta);\end{array} \frac{x\eta^{\eta}}{\tau^{\tau}s^{\lambda}}\right]. \tag{6.13}$$

Corollary 15. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau, \eta \in \mathbb{R}^+$, $\eta < \Re(\lambda) + \tau$, *then, for* $\Re(\nu) > 0, \Re(\vartheta) > 0$, *the following formula holds true:*

$$B\left\{E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}\left(xz^{\lambda}\right); \nu, \vartheta\right\} = \Gamma(\vartheta)E^{\gamma,\delta,\eta}_{\lambda,\nu+\vartheta,\tau}(x). \tag{6.14}$$

Corollary 16. *Let* $\lambda, \nu, \gamma, \delta \in \mathbb{C}$, $\min\{\Re(\lambda), \Re(\nu), \Re(\gamma), \Re(\delta)\} > 0$, $\tau, \eta \in \mathbb{R}^+$, $\eta < \Re(\lambda) + \tau$, *then, for* $\Re(l) > 0, \Re(\zeta) > 0$, *the following formula holds true:*

$$\begin{aligned}&\int_0^{\infty} z^{l-1}e^{-\zeta z/2}W_{a,b}(\zeta z)E^{\gamma,\delta,\eta}_{\lambda,\nu,\tau}\left(xz^{\vartheta}\right)dz\\ &\quad = \frac{\Gamma(\delta)\zeta^{-l}}{\Gamma(\gamma)}\,{}_4\Psi_3\left[\begin{array}{c}(\gamma,\eta), \left(\frac{1}{2}+b+l, \vartheta\right), \left(\frac{1}{2}-b+l, \vartheta\right), (1,1);\\ (\nu,\lambda), (\delta,\tau), \left(\frac{1}{2}-a+l, \vartheta\right);\end{array} \left(\frac{x}{\zeta^{\vartheta}}\right)\right].\end{aligned} \tag{6.15}$$

Competing interests

The author declares to have no competing interests.

Authors Contributions

All authors contribute equally in the present investigation.

7. Conclusion

In this work we give a new generalization of Mittag-Leffler function and studied elementary properties for the same. From the close relationship of the (p, k)-Mittag-Leffler function with many special functions, we can easily derive various known and new results.

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