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SOME RESULTS OF GENERALIZED LAGUERRE POLYNOMIALS WITH OTHER SPECIAL FUNCTIONS

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Abstract. This paper explores a generalized four-parameter class of Laguerre-type polynomials, denoted by $L_n^{\sigma,\rho}(\vartheta;x)$, this gives broaden traditional polynomial structures using advanced operational techniques. The investigation includes series expansions involving Hermite and Legendre polynomials, as well as composition formulae connected to generalized fractional integrals and derivatives. Moreover, integral transforms and several special cases have been discussed.

1. Introduction

In 2009, Shukla and Prajapati [**13**] introduced a class of generalized Laguerre-type polynomials denoted by $L_n^{\sigma,\rho}(\vartheta;x)$, defined as

$$L_n^{\sigma,\rho}(\vartheta;x) = \Gamma(\sigma n+\rho+1)\sum_{m=0}^{n}\frac{(-1)^m x^m}{m!\,\Gamma(\vartheta n-\vartheta m+1)\,\Gamma(\sigma m+\rho+1)}, \tag{1.1}$$

where $\vartheta,\sigma,\rho\in\mathbb{C}$ with $\Re(\vartheta)>0$, $\Re(\sigma)>0$, and $\Re(\rho)>-1$, and $n\in\mathbb{N}_0$.

Also, the generating function of (1.1) obtain by Shukla and Prajapati [**13**] as

$$\sum_{n=0}^{\infty}\frac{L_n^{\sigma,\rho}(\vartheta;x)t^n}{\Gamma(\sigma n+\rho+1)} = E_{\vartheta}(t)W(\sigma,\rho+1;-xt), \tag{1.2}$$

where,$E_{\vartheta}(t)$ is well-known Mittag-Leffler [**9**] function defined as

$$E_{\alpha}(t) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, z\in\mathbb{C},\alpha\geq 0. \tag{1.3}$$

and $W(\alpha,\beta+1;-z)$ is the Wright function [**17**] defined as

$$W(\vartheta,\sigma+1;-z) = \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(\vartheta n+\sigma+1)}. \tag{1.4}$$

Setting $\vartheta=1$ in (1.1) yields a subclass of polynomials previously studied by Prabhakar and Suman [**11**], defined by

$$L_n^{\sigma,\rho}(x) = \frac{\Gamma(\sigma n+\rho+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,x^k}{k!\,\Gamma(\sigma k+\rho+1)} \tag{1.5}$$

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Furthermore, taking $\sigma = 1$ in (1.1) reduces the polynomial to another class studied by Shukla and Prajapati [**10**], given by

$$L_n^{\rho}(\vartheta; x) = \sum_{m=0}^{n} \frac{(-1)^m (\rho+1)_n \, x^m}{m! \, \Gamma(\vartheta n - \vartheta m + 1) \, (\rho+1)_m} \tag{1.6}$$

The polynomials defined in (1.1), (1.5) and (1.6) have significant applications in the analytical solution of fractional-order differential equations and in spectral representations of operators [**7**].

The generalized hypergeometric function is defined (Rainville[**12**]) as,

$${}_pF_q \left[\begin{matrix} \mu_1, \ldots, \mu_p \\ \nu_1, \ldots, \nu_q \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(\mu_1)_k \cdots (\mu_p)_k}{(\nu_1)_k \cdots (\nu_q)_k} \frac{z^k}{k!} \tag{1.7}$$

The convergence of the series (1.7) is characterized as follows:

- If $p \leq q$, it converges for all finite z.
- If $p = q + 1$, it converges for $|z| < 1$ and diverges for $|z| > 1$.
- If $p > q + 1$, it diverges for all $z \neq 0$.
- For $p = q + 1$, it converges absolutely on $|z| = 1$ if

$$\Re \left(\sum_{j=1}^{q} \nu_j - \sum_{i=1}^{p} \mu_i \right) > 0.$$

Desai and Shukla [**1**] introduced the function ${}_pR_q(\tau, \mu; z)$, Thakkar and Shukla [**16**] studied further results of ${}_pR_q(\tau, \mu; z)$ function as

$${}_pR_q(\tau, \mu; z) = {}_pR_q \left[\begin{matrix} \vartheta_1, \vartheta_2, \ldots, \vartheta_p \\ \sigma_1, \sigma_2, \ldots, \sigma_q \end{matrix} \middle| \tau, \mu; z \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\tau k + \mu)} \frac{(\vartheta_1)_k \cdots (\vartheta_p)_k}{(\sigma_1)_k \cdots (\sigma_q)_k} \frac{z^k}{k!}, \tag{1.8}$$

where $p, q \in \mathbb{N}_0$ and $\tau, \mu \in \mathbb{C}, \Re(\tau), \Re(\mu), \Re(\vartheta_i), \Re(\sigma_j) > 0$ for any $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$.

In (1.8) the notation $(\nu)_k$ is a Pochhammer symbol defined (Rainville [**12**]) for $\nu \in \mathbb{C}$ by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} \nu(\nu+1) \cdots (\nu+k-1), & (k \in \mathbb{N}), \\ 1, & (k = 0, \nu \neq 0). \end{cases}$$

The series (1.8) is defined when no σ_j $(j = 1, 2, ..., q)$ is zero or a negative integer. If any numerator parameter ϑ_i $(i = 1, 2, ..., p)$ is a zero or negative integer, the series (1.8) terminates to polynomial in z.

The confluent hypergeometric function ${}_1F_1(a; c; z)$, also known as the Kummer function [**12**], is defined by

$${}_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.9}$$

the series (1.9) is converges absolutely for $|z| < \infty$.

The Series expansion of x^n in terms of terms of Hermite polynomials given in (Rainville [**12**]),

$$x^n = \frac{n!}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{n-2m}(x)}{m!(n-2m)!}. \tag{1.10}$$

The Series expansion of x^n in terms of terms of Legendre polynomials given in (Rainville [**12**]),

$$x^n = \frac{n!}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n-4m+1)P_{n-2m}(x)}{m!(\frac{3}{2})_{n-m}}. \tag{1.11}$$

The following series manipulation techniques as given in (Rainville [**12**]),

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A(m,n-m) \tag{1.12}$$

and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} B(m,n) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B(m,n+m) \tag{1.13}$$

Let $f \in C^n[a,b]$ and $\mu \in (n-1,n)$, where $n = \lceil \mu \rceil \in \mathbb{N}$. The *Caputo fractional derivative* [**4**] of order μ of the function f is defined as:

$${}^{C}D^{\mu}_{a+} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\mu-n+1}} \, d\tau, \tag{1.14}$$

for $t > 0$, where $\Gamma(\cdot)$ is the Gamma function and $f^{(n)}(\tau)$ is the n-th ordinary derivative of f.

Saigo [**14**] introduced the *left- and right-sided generalized fractional integrals*, defined respectively as

$$\left(I^{\sigma,\eta,\rho}_{0+} f\right)(x) = \frac{x^{-\sigma-\eta}}{\Gamma(\sigma)} \int_0^x (x-u)^{\sigma-1} \, {}_2F_1 \left[\begin{matrix} \sigma+\eta, -\rho; \\ \sigma; \end{matrix} \; 1 - \frac{u}{x} \right] f(u) \, du, \tag{1.15}$$

and

$$\left(I^{\sigma,\eta,\rho}_{-} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^{\infty} (u-x)^{\sigma-1} u^{-\sigma-\eta} \, {}_2F_1 \left[\begin{matrix} \sigma+\eta, -\rho; \\ \sigma; \end{matrix} \; 1 - \frac{u}{x} \right] f(u) \, du, \tag{1.16}$$

where $\sigma, \eta, \rho \in \mathbb{C}$, $\Re(\sigma) > 0$, and $x > 0$.

Special cases of these operators yield well-known fractional integrals:

- **Riemann–Liouville Fractional Integrals:** Setting $\eta = -\sigma$ in (1.15) and (1.16) reduces them to the classical left- and right-sided Riemann–Liouville fractional integrals [**15**]:

$$\left(I^{\sigma}_{0+} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_0^x (x-u)^{\sigma-1} f(u) \, du, \quad (x > 0, \; \Re(\sigma) > 0), \tag{1.17}$$

$$\left(I^{\sigma}_{-} f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^{\infty} (u-x)^{\sigma-1} f(u) \, du, \quad (x > 0, \; \Re(\sigma) > 0). \tag{1.18}$$

- **Erdélyi–Kober Fractional Integrals:** Setting $\eta = 0$ in (1.15) and (1.16) yields the Erdélyi–Kober fractional integrals [**15**], defined for $\sigma, \rho \in \mathbb{C}$ as:

$$\left(I_{0+}^{\sigma,\rho} f\right)(x) = \frac{x^{-\sigma-\rho}}{\Gamma(\sigma)} \int_0^x (x-u)^{\sigma-1} u^{\rho} f(u)\, du, \quad (x>0,\ \Re(\sigma)>0), \tag{1.19}$$

$$\left(I_{-}^{\sigma,\rho} f\right)(x) = \frac{x^{\rho}}{\Gamma(\sigma)} \int_x^\infty (u-x)^{\sigma-1} u^{-\sigma-\rho} f(u)\, du, \quad (x>0,\ \Re(\sigma)>0). \tag{1.20}$$

Following lemmas and a classical integral identity that will be useful for futher study.

Lemma 1.1 (Kilbas and Samko [**8**]). *Let $\sigma, \eta, \rho, \omega \in \mathbb{C}$ with $\Re(\sigma) > 0$ and $\Re(\omega) > \max\{0, \Re(\eta-\rho)\}$. Then,*

$$\left(I_{0+}^{\sigma,\eta,\rho} u^{\omega-1}\right)(x) = \frac{\Gamma(\omega)\Gamma(\omega+\rho-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\sigma+\rho)} x^{\omega-\eta-1}. \tag{1.21}$$

In particular, the following special cases hold (see Samko et al. [**15**]):

$$\left(I_{0+}^{\sigma} u^{\omega-1}\right)(x) = \frac{\Gamma(\omega)}{\Gamma(\omega+\sigma)} x^{\omega+\sigma-1}, \quad (\Re(\sigma)>0,\ \Re(\omega)>0), \tag{1.22}$$

$$\left(I_{0+}^{\sigma,\rho} u^{\omega-1}\right)(x) = \frac{\Gamma(\omega+\rho)}{\Gamma(\omega+\sigma+\rho)} x^{\omega-1}, \quad (\Re(\sigma)>0,\ \Re(\omega)>-\Re(\rho)). \tag{1.23}$$

Lemma 1.2 (Kilbas and Samko [**8**]). *Let $\sigma, \eta, \rho, \omega \in \mathbb{C}$ with $\Re(\sigma) > 0$ and $\Re(\omega) < 1 + \min\{\Re(\eta), \Re(\rho)\}$. Then,*

$$\left(I_{-}^{\sigma,\eta,\rho} u^{\omega-1}\right)(x) = \frac{\Gamma(\eta-\omega+1)\Gamma(\rho-\omega+1)}{\Gamma(1-\omega)\Gamma(\sigma+\eta+\rho-\omega+1)} x^{\omega-\eta-1}. \tag{1.24}$$

In particular, we have (see Samko et al. [**15**]):

$$\left(I_{-}^{\sigma} u^{\omega-1}\right)(x) = \frac{\Gamma(1-\sigma-\omega)}{\Gamma(1-\omega)} x^{\omega+\sigma-1}, \quad (0<\Re(\sigma)<1-\Re(\omega)), \tag{1.25}$$

$$\left(I_{-}^{\sigma,\rho} u^{\omega-1}\right)(x) = \frac{\Gamma(\rho-\omega+1)}{\Gamma(\sigma+\rho-\omega+1)} x^{\omega-1}, \quad (\Re(\omega)<1+\Re(\rho)). \tag{1.26}$$

Definition 1.3. [**3**] Let f be a real or complex-valued function defined on the interval $(0,1)$, and let $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$. The Beta transform of f, denoted by $\mathcal{B}\{f(t)\}$, is defined as

$$\mathcal{B}\{f(t); \vartheta, \sigma\} = \int_0^1 t^{\vartheta-1}(1-t)^{\sigma-1} f(t)\, dt. \tag{1.27}$$

Recently, Jatav and Shukla [**5, 6**] investigated various properties of a class of Laguerre-type polynomials $\mathcal{L}_n^{\delta,\xi}(x)$, originally introduced by Prabhakar and Suman. We appreciated their work and motivated by their contributions, this paper explores a more generalized three-parameter family of Laguerre polynomials $L_n^{\sigma,\rho}(\vartheta; x)$, introduced by Shukla and Prajapati [**13**], by employing methods from integral transforms, generalized fractional integration, and fractional differential operators. For foundational results and related developments refer to [**5, 6, 10, 11, 13**].
The structure of the paper is as follows: Sections 2–6 derive the series representations, generalized fractional integral operators , fractional derivative operator, and integral transform for $L_n^{\beta,\gamma}(\alpha; x)$.

2. Series Representation

Theorem 2.1 (Generalized Laguerre Polynomials with Hermite Polynomials)**.** *Let $\vartheta, \sigma, \rho \in \mathbb{C}$ with $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > -1$. Then, the following identity holds:*

$$L_n^{\sigma,\rho}(\vartheta; x) = \Gamma(\sigma n+\rho+1) \sum_{k=0}^{n} \sum_{m=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^k 2^{-k}}{m!(k-2m)!\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)} H_{k-2m}(x) \tag{2.1}$$

Proof. Starting from the generating function:

$$\sum_{n=0}^{\infty} \frac{L_n^{\sigma,\rho}(\vartheta; x)}{\Gamma(\sigma n+\rho+1)} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k x^k t^n}{k!\,\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)}.$$

Using the Hermite polynomial expansion (1.10), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{L_n^{\sigma,\rho}(\vartheta; x)}{\Gamma(\sigma n+\rho+1)} t^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k t^n}{k!\,\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)} \cdot x^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k t^n}{k!\,\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)} \cdot \left[2^{-k} \sum_{m=0}^{\left[\frac{k}{2}\right]} \frac{k!}{m!(k-2m)!} H_{k-2m}(x) \right] \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^k 2^{-k} t^n}{m!(k-2m)!\,\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)} H_{k-2m}(x).
\end{aligned}$$

Now, compare the coefficients of t^n on both sides to extract:

$$\frac{L_n^{\sigma,\rho}(\vartheta; x)}{\Gamma(\sigma n+\rho+1)} = \sum_{k=0}^{n} \sum_{m=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^k 2^{-k}}{m!(k-2m)!\,\Gamma(\vartheta(n-k)+1)\Gamma(\sigma k+\rho+1)} H_{k-2m}(x).$$

□

Substituting $\sigma = 1$ into the identity (2.1) yields the following corollary.

Corollary 2.2. *Let $\vartheta, \rho \in \mathbb{C}$ with $\Re(\vartheta) > 0$ and $\Re(\rho) > -1$. Then, the following identity holds:*

$$L_n^{\rho}(\vartheta; x) = \Gamma(\rho+n+1) \sum_{k=0}^{n} \sum_{m=0}^{\left\lfloor\frac{k}{2}\right\rfloor} \frac{(-1)^k 2^{-k}}{m!(k-2m)!\Gamma(\vartheta(n-k)+1)\Gamma(\rho+k+1)} H_{k-2m}(x) \tag{2.2}$$

Theorem 2.3 (Generalized Laguerre Polynomials with Legendre Polynomials)**.** *Let $\vartheta, \sigma, \rho \in \mathbb{C}$ with $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > -1$. Then, the following* *identity holds:*

$$L_n^{\sigma,\rho}(\vartheta;x) = \Gamma(\sigma n+\rho+1)\sum_{k=0}^{n}\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2k+1)P_k(x)}{2^{k+2m}m!\left(\frac{3}{2}\right)_{k+m}} \times \frac{1}{\Gamma(\vartheta(n-k-2m)+1)\,\Gamma(\sigma(k+2m)+\rho+1)} \tag{2.3}$$

Proof. Starting from the generating function (1.1), we have

$$\sum_{n=0}^{\infty}\frac{L_n^{\sigma,\rho}(\vartheta;x)t^n}{\Gamma(\sigma n+\rho+1)} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k x^k t^n}{k!\,\Gamma(\vartheta(n-k)+1)\,\Gamma(\sigma k+\rho+1)} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^{n+k}}{k!\,\Gamma(\vartheta n+1)\,\Gamma(\sigma k+\rho+1)}$$

Next, applying the Legendre expansion (1.11), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{L_n^{\sigma,\rho}(\vartheta;x)t^n}{\Gamma(\sigma n+\rho+1)}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\left[\frac{k}{2}\right]}\frac{(-1)^k(2k-4m+1)P_{k-2m}(x)t^{n+k}}{2^k m!\left(\frac{3}{2}\right)_{k-m}\Gamma(\vartheta n+1)\Gamma(\sigma k+\rho+1)}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^k(2k+1)P_k(x)t^{n+k+2m}}{2^{k+2m}m!\left(\frac{3}{2}\right)_{k+m}\Gamma(\vartheta n+1)\Gamma(\sigma(k+2m)+\rho+1)}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2k+1)P_k(x)t^{n+k}}{2^{k+2m}m!\left(\frac{3}{2}\right)_{k+m}\Gamma(\vartheta(n-2m)+1)\,\Gamma(\sigma(k+2m)+\rho+1)}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2k+1)P_k(x)t^{n}}{2^{k+2m}m!\left(\frac{3}{2}\right)_{k+m}\Gamma(\vartheta(n-k-2m)+1)\,\Gamma(\sigma(k+2m)+\rho+1)}
\end{aligned}$$

Comparing the coefficients of t^n on both sides yields the desired result (2.3). □

Substituting $\sigma = 1$ into the identity (2.3) yields the following corollary.

Corollary 2.4. *Let $\vartheta,\rho\in\mathbb{C}$ with $\Re(\vartheta)>0$ and $\Re(\rho)>-1$. Then, the following identity holds:*

$$L_n^{\rho}(\vartheta;x) = \Gamma(\rho+n+1) \times\sum_{k=0}^{n}\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2k+1)P_k(x)}{2^{k+2m}m!\left(\frac{3}{2}\right)_{k-m}\Gamma(\vartheta(n-k-2m)+1)\Gamma(\rho+k+2m+1)} \tag{2.4}$$

3. Generalized Fractional Integration of Polynomials $L_n^{\sigma,\rho}(\vartheta;x)$

In this section, we compute the left and right sided generalized fractional integration with the polynomials $L_n^{\sigma,\rho}(\vartheta;x)$, expressed in terms of Mittag-Leffler function and ${}_pR_q$ function.

Theorem 3.1. *Let $\alpha,\eta,\gamma,\omega,\sigma,\rho,\vartheta \in \mathbb{C}$ such that $\Re(\alpha)>0$, $\Re(\sigma)>0$, $\Re(\vartheta)>0$, $\Re(\rho)>-1$, $\Re(\omega)>\max\{0,\Re(\eta-\gamma)\}$. Then*

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}t^n\right)(x)=x^{\omega-\eta-1}\frac{\Gamma(\omega)\Gamma(\omega+\gamma-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)}\,E_{\vartheta}(t) \times {}_2R_2\left[\begin{matrix}\omega,\omega+\gamma-\eta\\ \omega-\eta,\omega+\alpha+\gamma\end{matrix}\middle|\sigma,\rho+1;-x\right] \tag{3.1}$$

Proof. Consider the left-hand side of (3.1)

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}t^n\right)(x)$$
$$=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^x(x-u)^{\alpha-1}{}_2F_1\left[\begin{matrix}\alpha+\eta,-\gamma;\\ \alpha;\end{matrix}\ 1-\frac{u}{x}\right]\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}t^n\,du$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^kt^n}{\Gamma(\vartheta n-\vartheta k+1)\Gamma(\sigma k+\rho+1)k!}\left(I_{0+}^{\alpha,\eta,\gamma}u^{\omega+k-1}\right)(x)$$

For any $k=0,1,2,...,\Re(\omega+k)\geq\Re(\omega)>max[0,\Re(\eta-\gamma)]$. From Lemma (1.1), further using (1.21) and replacing ω by $\omega+k$, we get

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}t^n\right)(x)$$
$$=x^{\omega-\eta-1}$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k\Gamma(\omega+k)\Gamma(\omega+\gamma-\eta+k)x^kt^n}{\Gamma(\vartheta n-\vartheta k+1)\Gamma(\sigma k+\rho+1)\Gamma(\omega-\eta+k)\Gamma(\omega+\alpha+\gamma+k)k!}$$
$$=x^{\omega-\eta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\omega)\Gamma(\omega+\gamma-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)\Gamma(\vartheta n+1)} \tag{3.2}$$
$$\times\sum_{k=0}^{\infty}\frac{(\omega)_k(\omega+\gamma-\eta)_k(-x)^kt^{n+k}}{\Gamma(\sigma k+\rho+1)(\omega-\eta)_k(\omega+\alpha+\gamma)_kk!}$$
$$=x^{\omega-\eta-1}\frac{\Gamma(\omega)\Gamma(\omega+\gamma-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)}\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\vartheta n+1)}$$
$$\times\sum_{k=0}^{\infty}\frac{(\omega)_k(\omega+\gamma-\eta)_k(-x)^kt^k}{\Gamma(\sigma k+\rho+1)(\omega-\eta)_k(\omega+\alpha+\gamma)_kk!},$$

on using (1.8), we obtain

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}t^n\right)(x)=x^{\omega-\eta-1}\frac{\Gamma(\omega)\Gamma(\rho+1)\Gamma(\omega+\gamma-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)}\,E_{\vartheta}(t)$$
$$\times\,{}_2R_2\left[\begin{matrix}\omega,\omega+\gamma-\eta\\ \omega-\eta,\omega+\alpha+\gamma\end{matrix}\middle|\sigma,\rho+1;-x\right],$$

this completes the proof. □

On setting $\eta = -\alpha$ and $\eta = 0$ in above theorem, we arrive at the following corollaries.

Corollary 3.2. *Let $\alpha, \omega, \sigma, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) > 0$. Then*

$$\left(I_{0+}^{\alpha}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}\right)(x) = x^{\omega+\alpha-1}\frac{\Gamma(\omega)\Gamma(\rho+1)}{\Gamma(\omega+\alpha)} \times E_{\vartheta}(t)\,{}_1R_1\left[\begin{matrix}\omega\\ \omega+\alpha\end{matrix}\,\middle|\sigma,\rho+1;-x\right] \tag{3.3}$$

Corollary 3.3. *Let $\alpha, \gamma, \omega, \sigma, \vartheta, \rho \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) > -\Re(\gamma)$. Then*

$$\left(I_{0+}^{\alpha,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}(\vartheta;u)}{\Gamma(\sigma n+\rho+1)}\right)(x) = x^{\omega-1}\frac{\Gamma(\omega+\gamma)\Gamma(\rho+1)}{\Gamma(\omega+\alpha+\gamma)} \times E_{\vartheta}(t)\,{}_1R_1\left[\begin{matrix}\omega+\gamma\\ \omega+\alpha+\gamma\end{matrix}\,\middle|\sigma,\rho+1;-x\right] \tag{3.4}$$

Theorem 3.4. *Let $\alpha, \eta, \gamma, \omega, \sigma, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) < 1 + \min\{\Re(\eta), \Re(\gamma)\}$. Then*

$$\left(I_{-}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}\left(\vartheta;\frac{1}{u}\right)t^n}{\Gamma(\sigma n+\rho+1)}\right)(x) = x^{\omega-\eta-1}\frac{\Gamma(\eta-\omega+1)\Gamma(\gamma-\omega+1)}{\Gamma(1-\omega)\Gamma(\alpha+\eta+\gamma-\omega+1)} \times E_{\vartheta}(t)\,{}_2R_2\left[\begin{matrix}\eta-\omega+1,\gamma-\omega+1\\ 1-\omega,\alpha+\eta+\gamma-\omega+1\end{matrix}\,\middle|\sigma,\rho+1;-\frac{1}{x}\right] \tag{3.5}$$

Proof. The left hand side of (3.5), gives

$$\begin{aligned}&\left(I_{-}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}\left(\vartheta;\frac{1}{u}\right)t^n}{\Gamma(\sigma n+\rho+1)}\right)(x)\\ &= \frac{1}{\Gamma(\alpha)}\int_x^{\infty}(u-x)^{\alpha-1}u^{-\alpha-\eta}\,{}_2F_1\left[\begin{matrix}\alpha+\eta,-\gamma;\\ \alpha;\end{matrix}\ 1-\frac{u}{x}\right]\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\sigma,\rho}\left(\vartheta;\frac{1}{u}\right)t^n}{\sigma n+\rho+1}du\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k t^n}{\Gamma(\vartheta n-\vartheta k+1)\Gamma(\sigma k+\rho+1)k!}\left(I_{-}^{\alpha,\eta,\gamma}u^{\omega-k-1}\right)(x)\end{aligned}$$

For any $k = 0, 1, 2, ..., \Re(\omega - k) < \Re(\omega) < 1 + \min\{\Re(\eta), \Re(\gamma)\}$. From Lemma 1.2, further using (1.24) and replacing ω by $\omega - k$, we get

$$\begin{aligned}
&\left(I_{-}^{\alpha,\eta,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\sigma,\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\sigma n + \rho + 1)}\right)(x) \\
&= x^{\omega-\eta-1} \\
&\quad \times \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k \Gamma(\eta - \omega + 1 + k)\Gamma(\gamma - \omega + 1 + k) x^{-k}}{\Gamma(\vartheta n - \vartheta k + 1)\Gamma(\sigma k + \rho + 1)\Gamma(1 - \omega + k)\Gamma(\alpha + \eta + \gamma - \omega + 1 + k) k!} \\
&= x^{\omega-\eta-1} \sum_{n=0}^{\infty} \frac{\Gamma(\eta - \omega + 1)\Gamma(\gamma - \omega + 1) t^n}{\Gamma(1 - \omega)\Gamma(\alpha + \eta + \gamma - \omega + 1)\Gamma(\vartheta n + 1)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(\eta - \omega + 1)_k (\gamma - \omega + 1)_k (-x)^{-k} t^k}{\Gamma(\sigma k + \rho + 1)(1 - \omega)_k (\alpha + \eta + \gamma - \omega + 1)_k k!} \\
&= x^{\omega-\eta-1} \frac{\Gamma(\eta - \omega + 1)\Gamma(\gamma - \omega + 1)}{\Gamma(1 - \omega)\Gamma(\alpha + \eta + \gamma - \omega + 1)} \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\vartheta n + 1)} \\
&\quad \times \sum_{k=0}^{\infty} \frac{(\eta - \omega + 1)_k (\gamma - \omega + 1)_k (-x)^{-k} t^k}{\Gamma(\sigma k + \rho + 1)(1 - \omega)_k (\alpha + \eta + \gamma - \omega + 1)_k k!}
\end{aligned} \tag{3.6}$$

On using (1.8) in the above equation, we get

$$\left(I_{-}^{\alpha,\eta,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\sigma,\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\sigma n + \rho + 1)}\right)(x) = x^{\omega-\eta-1} \frac{\Gamma(\eta - \omega + 1)\Gamma(\gamma - \omega + 1)}{\Gamma(1 - \omega)\Gamma(\alpha + \eta + \gamma - \omega + 1)} E_\vartheta(t)$$
$$\times\, {}_2R_2\left[\begin{array}{c} \eta - \omega + 1, \gamma - \omega + 1 \\ 1 - \omega, \alpha + \eta + \gamma - \omega + 1 \end{array} \middle| \sigma, \rho + 1; -\frac{t}{x}\right],$$

this completes the proof. □

On setting $\eta = -\alpha$ and $\eta = 0$ in the above theorem yields the following corollaries.

Corollary 3.5. *Let $\alpha, \omega, \sigma, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $0 < \Re(\alpha) < 1 - \Re(\omega)$. Then*

$$\begin{aligned}
\left(I_{-}^{\alpha} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\sigma,\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\sigma n + \rho + 1)}\right)(x) &= x^{\omega+\alpha-1} \frac{\Gamma(1 - \alpha - \omega)}{\Gamma(1 - \omega)} E_\vartheta(t) \\
&\quad \times\, {}_1R_1\left[\begin{array}{c} 1 - \alpha - \omega \\ 1 - \omega \end{array} \middle| \sigma, \rho + 1; -\frac{t}{x}\right]
\end{aligned} \tag{3.7}$$

Corollary 3.6. *Let $\alpha, \gamma, \omega, \sigma, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\sigma) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) < 1 + \Re(\gamma)$. Then*

$$\begin{aligned}
\left(I_{-}^{\alpha,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\sigma,\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\sigma n + \rho + 1)}\right)(x) &= x^{\omega-1} \frac{\Gamma(\gamma - \omega + 1)}{\Gamma(\alpha + \gamma - \omega + 1)} E_\vartheta(t) \\
&\quad \times\, {}_1R_1\left[\begin{array}{c} \gamma - \omega + 1 \\ \alpha + \gamma - \omega + 1 \end{array} \middle| \sigma, \rho + 1; -\frac{t}{x}\right]
\end{aligned} \tag{3.8}$$

4. Generalized Fractional Integration of Polynomials $L_n^{\rho}(\vartheta;x)$

In this section, we compute the left and right sided generalized fractional integration with the polynomials $L_n^{\rho}(\vartheta;x)$, expressed in terms of Mittag-Leffler function and Generalized Hypergeometric function.

Theorem 4.1. *Let* $\alpha,\eta,\gamma,\omega,\rho,\vartheta \in \mathbb{C}$ *such that* $\Re(\alpha)>0$, $\Re(\vartheta)>0$, $\Re(\rho)>-1$, $\Re(\omega)>\max\{0,\Re(\eta-\gamma)\}$. *Then*

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\rho}(\vartheta;u)t^n}{\Gamma(\rho+n+1)}\right)(x)=x^{\omega-\eta-1}\frac{\Gamma(\omega)\Gamma(\omega+\gamma-\eta)}{\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)\Gamma(\rho+1)}\,E_{\vartheta}(t)\times{}_2F_3\left[\begin{matrix}\omega,\omega+\gamma-\eta;\\ \rho+1,\omega-\eta,\omega+\alpha+\gamma;\end{matrix}\; -xt\right] \tag{4.1}$$

Proof. On applying theorem 3.1 and using (3.2) with $\sigma=1$, we get

$$\left(I_{0+}^{\alpha,\eta,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\rho}(\vartheta;u)t^n}{\Gamma(\rho+n+1)}\right)(x) = x^{\omega-\eta-1}\frac{\Gamma(\omega)\Gamma(\omega+\gamma-\eta)}{\Gamma(\rho+1)\Gamma(\omega-\eta)\Gamma(\omega+\alpha+\gamma)}\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\vartheta n+1)} \times\sum_{k=0}^{\infty}\frac{(\omega)_k(\omega+\gamma-\eta)_k(-xt)^k}{(\rho+1)_k(\omega-\eta)_k(\omega+\alpha+\gamma)_k k!}. \tag{4.2}$$

On using (1.7), this yeilds the result in (4.1). □

On setting $\eta=-\alpha$ and $\eta=0$ in above theorem yields the following corollaries.

Corollary 4.2. *Let* $\alpha,\omega,\rho,\vartheta \in \mathbb{C}$ *such that* $\Re(\alpha)>0$, $\Re(\vartheta)>0$, $\Re(\rho)>-1$, $\Re(\omega)>0$. *Then*

$$\left(I_{0+}^{\alpha}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\rho}(\vartheta;u)t^n}{\Gamma(\rho+n+1)}\right)(x)=x^{\omega+\alpha-1}\frac{\Gamma(\omega)}{\Gamma(\rho+1)\Gamma(\omega+\alpha)}\times E_{\vartheta}(t)\,{}_1F_2\left[\begin{matrix}\omega;\\ \rho+1,\omega+\alpha;\end{matrix}\; -xt\right] \tag{4.3}$$

Corollary 4.3. *Let* $\alpha,\gamma,\omega,\vartheta,\rho \in \mathbb{C}$ *such that* $\Re(\alpha)>0$, $\Re(\vartheta)>0$, $\Re(\rho)>-1$, $\Re(\omega)>-\Re(\gamma)$. *Then*

$$\left(I_{0+}^{\alpha,\gamma}\sum_{n=0}^{\infty}\frac{u^{\omega-1}L_n^{\rho}(\vartheta;u)t^n}{\Gamma(\rho+n+1)}\right)(x)=x^{\omega-1}\frac{\Gamma(\omega+\gamma)}{\Gamma(\rho+1)\Gamma(\omega+\alpha+\gamma)}\times E_{\vartheta}(t)\,{}_1F_2\left[\begin{matrix}\omega+\gamma;\\ \rho+1,\omega+\alpha+\gamma;\end{matrix}\; -xt\right] \tag{4.4}$$

Theorem 4.4. *Let $\alpha, \eta, \gamma, \omega, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) < 1 + \min\{\Re(\eta), \Re(\gamma)\}$. Then*

$$\left(I_{-}^{\alpha,\eta,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\rho+n+1)}\right)(x) = x^{\omega-\eta-1} \frac{\Gamma(\eta-\omega+1)\Gamma(\gamma-\omega+1)}{\Gamma(\rho+1)\Gamma(1-\omega)\Gamma(\alpha+\eta+\gamma-\omega+1)} \times E_{\vartheta}(t)\, {}_2F_3\left[\begin{matrix} \eta-\omega+1, \gamma-\omega+1 \\ \rho+1, 1-\omega, \alpha+\eta+\gamma-\omega+1 \end{matrix}\; -\frac{t}{x}\right] \tag{4.5}$$

Proof. On applying theorem 3.4 and using (3.6) with $\sigma = 1$, we get

$$\left(I_{-}^{\alpha,\eta,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\rho+n+1)}\right)(x)$$
$$= x^{\omega-\eta-1} \frac{\Gamma(\eta-\omega+1)\Gamma(\gamma-\omega+1)}{\Gamma(\rho+1)\Gamma(1-\omega)\Gamma(\alpha+\eta+\gamma-\omega+1)} \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\vartheta n+1)}$$
$$\times \sum_{k=0}^{\infty} \frac{(\eta-\omega+1)_k(\gamma-\omega+1)_k(-x)^{-k} t^k}{(\rho+1)_k(1-\omega)_k(\alpha+\eta+\gamma-\omega+1)_k k!}$$

using (1.7) yields (4.5). □

On setting $\eta = -\alpha$ and $\eta = 0$ in above theorem, we arrive at the following corollaries.

Corollary 4.5. *Let $\alpha, \omega, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $0 < \Re(\alpha) < 1 - \Re(\omega)$. Then*

$$\left(I_{-}^{\alpha} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\rho+n+1)}\right)(x) = x^{\omega+\alpha-1} \frac{\Gamma(1-\alpha-\omega)}{\Gamma(\rho+1)\Gamma(1-\omega)} \times E_{\vartheta}(t)\, {}_1F_2\left[\begin{matrix} 1-\alpha-\omega; \\ \rho+1, 1-\omega; \end{matrix}\; -\frac{t}{x}\right] \tag{4.6}$$

Corollary 4.6. *Let $\alpha, \gamma, \omega, \rho, \vartheta \in \mathbb{C}$ such that $\Re(\alpha) > 0$, $\Re(\vartheta) > 0$, $\Re(\rho) > -1$, $\Re(\omega) < 1 + \Re(\gamma)$. Then*

$$\left(I_{-}^{\alpha,\gamma} \sum_{n=0}^{\infty} \frac{u^{\omega-1} L_n^{\rho}\left(\vartheta; \frac{1}{u}\right) t^n}{\Gamma(\rho+n+1)}\right)(x) = x^{\omega-1} \frac{\Gamma(\gamma-\omega+1)}{\Gamma(\rho+1)\Gamma(\alpha+\gamma-\omega+1)} E_{\vartheta}(t) \times {}_1F_2\left[\begin{matrix} \gamma-\omega+1; \\ \rho+1, \alpha+\gamma-\omega+1; \end{matrix}\; -\frac{t}{x}\right] \tag{4.7}$$

5. Fractional Derivative Operator

In this section, we compute the Caputo fractional derivative operator of polynomials $L_n^{\sigma,\rho}(\vartheta; x)$.

Theorem 5.1. *Let $m \in \mathbb{N}$, $\sigma, \rho, \vartheta, \omega \in \mathbb{C}$ with $\Re(\omega) > 0$, $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > -1$. Then*

$$\left(\frac{d}{dx}\right)^m [x^\rho L_n^{\sigma,\rho}(\vartheta; \omega x^\sigma)] = x^{\rho-m} \frac{\Gamma(\sigma n + \rho + 1)}{\Gamma(\sigma n + \rho - m + 1)} L_n^{\sigma,\rho-m}(\vartheta; \omega x^\sigma). \tag{5.1}$$

Proof. Consider, the left hand side of (5.1)

$$\begin{aligned}
&\left(\frac{d}{dx}\right)^m [x^\rho L_n^{\sigma,\rho}(\vartheta; \omega x^\sigma)] \\
&= \left(\frac{d}{dx}\right)^m \left[\Gamma(\sigma n + \rho + 1) \sum_{k=0}^{n} \frac{(-1)^k (\omega)^k x^{\sigma k + \rho}}{\Gamma(\vartheta n - \vartheta k + 1)\Gamma(\sigma k + \rho + 1) k!}\right] \\
&= x^{\rho-m} \Gamma(\sigma n + \rho + 1) \sum_{k=0}^{n} \frac{(-1)^k (\omega x^\sigma)^k}{\Gamma(\vartheta n - \vartheta k + 1)\Gamma(\sigma k + \rho - m + 1) k!}
\end{aligned}$$

this immediately leads to form of (5.1). □

Theorem 5.2. *Let $m \in \mathbb{N}$, and let $\vartheta, \sigma, \rho, \omega, \mu \in \mathbb{C}$ such that $\Re(\omega) > 0$, $\Re(\mu) > 0$, $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$, and $\Re(\rho) > -1$. Then*

$$\left({}^C D_{0+}^{\mu} [u^\rho L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)]\right)(x) = x^{\rho-\mu} \frac{\Gamma(\sigma n + \rho + 1)}{\Gamma(\sigma n + \rho - \mu + 1)} L_n^{\sigma,\rho-\mu}(\vartheta; \omega x^\sigma), \tag{5.2}$$

where ${}^C D_{0+}^{\mu}$ denotes the Caputo fractional derivative of order μ.

Proof. Consider the left-hand side of equation (5.2):

$$\begin{aligned}
&\left({}^C D_{0+}^{\mu} [u^\rho L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)]\right)(x) \\
&\quad = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-u)^{m-\mu-1} \left(\frac{d}{du}\right)^m [u^\rho L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)]\, du.
\end{aligned}$$

Substituting the definition of $L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)$ from equation (1.1), we obtain

$$\begin{aligned}
&\left({}^C D_{0+}^{\mu} [u^\rho L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)]\right)(x) \\
&\quad = \Gamma(\sigma n + \rho + 1) \sum_{k=0}^{n} \frac{(-1)^k \omega^k}{k!\, \Gamma(\vartheta n - \vartheta k + 1)\, \Gamma(\sigma k + \rho - m + 1)} \\
&\quad \times \frac{1}{\Gamma(m-\mu)} \int_0^x (x-u)^{m-\mu-1} u^{\sigma k + \rho - m}\, du.
\end{aligned}$$

On using (1.17), we get

$$\begin{aligned}
&\left({}^C D_{0+}^{\mu} [u^\rho L_n^{\sigma,\rho}(\vartheta; \omega u^\sigma)]\right)(x) \\
&= \sum_{k=0}^{n} \frac{(-1)^k \omega^k \Gamma(\sigma n + \rho + 1)}{k!\, \Gamma(\vartheta n - \vartheta k + 1)\, \Gamma(\sigma k + \rho - m + 1)} \left(I_{0+}^{m-\mu} u^{\sigma k + \rho - m}\right)(x).
\end{aligned}$$

Using (1.22) in the preceding equation immediately leads to the form of (5.2). □

Substituting $\sigma = 1$ into the identity (5.2) yields the following corollary.

Corollary 5.3. *Let $m \in \mathbb{N}$, and let $\vartheta, \rho, \omega, \mu \in \mathbb{C}$ such that $\Re(\omega) > 0$, $\Re(\mu) > 0$, $\Re(\vartheta) > 0$, and $\Re(\rho) > -1$. Then*

$$\left({}^{C}D_{0+}^{\mu}\left[u^{\rho} L_n^{\rho}(\vartheta;\omega u)\right]\right)(x) = x^{\rho-\mu}\frac{\Gamma(\rho+n+1)}{\Gamma(\rho-\mu+n+1)} L_n^{\rho-\mu}(\vartheta;\omega x), \tag{5.3}$$

where ${}^{C}D_{0+}^{\mu}$ denotes the Caputo fractional derivative of order μ.

6. Beta Transforms

Theorem 6.1. *Let $\vartheta, \sigma, \rho, \kappa, \omega \in \mathbb{C}$ such that $\Re(\vartheta) > 0$, $\Re(\sigma) > 0$,$\Re(\kappa) > 0$, $\Re(\omega) > 0$, $\Re(\rho) > -1$. Then the Beta transform of the generalized Laguerre-type polynomial $L_n^{\sigma,\rho}(\vartheta;x)$ satisfies:*

$$\mathcal{B}[L_n^{\sigma,\rho}(\vartheta;\omega u^{\sigma});\rho+1,\kappa] = \frac{\Gamma(\kappa)\Gamma(\sigma n+\rho+1)}{\Gamma(\sigma n+\rho+\kappa+1)}\; L_n^{\sigma,\rho+\kappa}(\vartheta;\omega) \tag{6.1}$$

Proof. Consider the left hand side of (6.1) and applying (1.27),

$$\begin{aligned}
&\mathcal{B}[L_n^{\sigma,\rho}(\vartheta;\omega u^{\sigma});\rho+1,\kappa]\\
&= \int_0^1 u^{\rho}(1-u)^{\kappa-1} L_n^{\sigma,\rho}(\vartheta;\omega u^{\sigma})\,du\\
&= \Gamma(\sigma n+\rho+1)\sum_{k=0}^{n}\frac{(-1)^k\omega^k}{k!\,\Gamma(\vartheta n-\vartheta k+1)\,\Gamma(\sigma k+\rho+1)}\int_0^1 u^{\sigma k+\rho}(1-u)^{\kappa-1}\,du\\
&= \Gamma(\kappa)\Gamma(\sigma n+\rho+1)\sum_{k=0}^{n}\frac{(-1)^k\omega^k}{k!\,\Gamma(\vartheta n-\vartheta k+1)\,\Gamma(\sigma k+\rho+\kappa+1)},
\end{aligned}$$

applying equation (1.1) to the preceding expression yields the desired result in (6.1). □

In particular, setting $\kappa = \frac{1}{2}$ in equation (6.1) yields

$$\mathcal{B}\left[L_n^{\sigma,\rho}(\vartheta;\omega u^{\sigma});\rho+1,\frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\sigma n+\rho+1)}{\Gamma(\sigma n+\rho+\frac{3}{2})}\; L_n^{\sigma,\rho+\frac{1}{2}}(\vartheta;\omega) \tag{6.2}$$

Substituting $\sigma = 1$ into the identity (6.1) yields the following corollary.

Corollary 6.2. *Let $\vartheta, \rho, \kappa, \omega \in \mathbb{C}$ such that $\Re(\vartheta) > 0$, $\Re(\kappa) > 0$, $\Re(\omega) > 0$, $\Re(\rho) > -1$. Then the Beta transform of the generalized Laguerre-type polynomial $L_n^{\rho}(\vartheta;x)$ satisfies:*

$$\mathcal{B}[L_n^{\rho}(\vartheta;\omega u^{\sigma});\rho+1,\kappa] = \frac{\Gamma(\kappa)\Gamma(\rho+n+1)}{\Gamma(\rho+\kappa+n+1)}\; L_n^{\rho+\kappa}(\vartheta;\omega) \tag{6.3}$$

In particular, setting $\kappa = \frac{1}{2}$ in equation (6.3) yields

$$\mathcal{B}\left[L_n^{\rho}(\vartheta;\omega u);\rho+1,\frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma(\rho+n+1)}{\Gamma(\rho+n+\frac{3}{2})}\; L_n^{\rho+\frac{1}{2}}(\vartheta;\omega) \tag{6.4}$$

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