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Semigoup theoretic properties of Idempotents in regular rings

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Abstract

A ring is a semigroup under the multiplication in the ring. The properties of semigroups can be carried over to rings. This give more new results as there is an addition in rings.
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1 Introduction

A *semigroup*, by definition, is a set with an associative binary operation. Thus every group is a semigroup and every ring is a semigroup with respect to its multiplication. In this chapter, we see how certain concepts and techniques used in the theory of semigroups can be applied to general rings. The results obtained here are refined for special classes of rings in later chapters. Since the theory of rings has a long history, while the theory of semigroups is relatively new, we assume many of the ideas about ring theory to be well known, but give detailed descriptions of ideas about semigroups.

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Notations and terminology of rings are as in [2] and those of semigroups as in [1], unless otherwise specified. The composite of two elements in a semigroup is called a product, and the product of two elements x, y in a semigroup or ring is written xy. For subsets X and Y of a semigroup or ring, we write XY for the set of all products xy with x in X and y in Y. Again, repeated product of a single element with itself is written as exponentiation, such as x^2 for xx.

2 Ideals and idempotents

Ideals of a ring play an important role in elucidating the structure of the whole ring. Recall that a subset I of a ring R is called a left ideal, if I is a subring of R such that $RI \subseteq I$. Dually I is called a *right ideal*, if I is a subring of R such that $IR \subseteq I$. These definitions can be extended to semigroups: a subset I of a semigroup S is called a left ideal if $SI \subseteq I$, and a right ideal if $IS \subseteq S$, the product xy is in I.

For an element a of a semigroup or a ring, the intersection set of all left (right) ideals containing a is the smallest left (right) ideal containing a and is called the *principal left* (*right*) *ideal* generated by a. The principal left ideal generated by an element a of a semigroup S can be easily seen to be $Sa \cup \{a\}$ and we denote it by S^1a; dually, the principal right ideal generated by a is $aS \cup \{a\}$, and we denote it by aS^1. Note that if S contains a multiplicative identity, then $S^1a = Sa$ and $aS^1 = aS$.

Equality of principal left ideals or right ideals are equivalence relations on a semigroup. They are denoted by $\mathscr{L}$ and $\mathscr{R}$. More precisely, we define

$$\mathscr{L} = \{(x, y) \in S \times S \colon S^1x = S^1y\}$$
$$\mathscr{R} = \{(x, y) \in S \times S \colon xS^1 = yS^1\}$$

Since these are equivalence relations on S, each of them partitions S into equivalence classes. We define the $\mathscr{L}$-class and the $\mathscr{R}$-class of x in S by

$$L_x = \{y \in S \colon y \mathrel{\mathscr{L}} x\}$$
$$R_x = \{y \in S \colon y \mathrel{\mathscr{R}} x\}$$

Now an element e of S is said to be an *idempotent*, if $e^2 = e$. The $\mathscr{L}$-class and $\mathscr{R}$-class of an idempotent can be described as follows (cf.[1], Lemma 2.14):

Proposition 2.1. *Let e be an idempotent in a semigroup S. Then we have the following:*

(i) *$x \in L_e$ if and only if $xe = x$ and $x'x = e$ for some x' in S*

(ii) $x \in R_e$ *if and only if* $ex = x$ *and* $xx' = e$ *for some* x' *in* S

Proof. First suppose that $x \in L_e$. Then $x \mathscr{L} e$, so that $S^1x = S^1e$, by definition. Since $x \in S^1x = S^1e$, we have $x = ye$ for some y in S^1, so that $xe = ye^2 = ye = x$. Again since $e \in S^1e = S^1x = Sx \cup \{x\}$, we have $e = x$ or $e \in Sx$; if $e = x$, then taking $x' = e$, we get $x'x = e^2 = e = x$ and if $e \in Sx$, then there exists x' in S with $x'x = e$.

Conversely suppose that $xe = x$ and $x'x = e$ for some x' in S. Then $S^1x = S^1xe \subseteq S^1e$ and $S^1e = S^1x'x \subseteq S^1x$. Thus $S^1x = S^1e$ and so $x \mathscr{L} e$, by definition and hence $x \in L_e$ This proves (i). A dual argument proves (ii) □

Thus idempotents in an $\mathscr{L}$-class are right identities for the elements of the class and each element in the class has left inverses with respect to each of these idempotents. The dual of this is the case for $\mathscr{R}$-classes. So if an element of a semigroup is both $\mathscr{L}$-related and $\mathscr{R}$-related to an idempotent e, then e is right identity for x and x has a right inverse. This motivates the consideration of the intersection of $\mathscr{L}$ and $\mathscr{R}$, which is also an equivalence relation on S. We define

$$\mathscr{H} = \mathscr{L} \bigcap \mathscr{R}$$

and for each x in S,

$$H_x = \{y \in S \colon y \mathscr{H} x\}$$

Thus if e is an idempotent in S, then e is a right (and left) identity for all elements of H_e and each element of H_e has a right (and left) inverse with respect to e. Thus we have the following (cf.[1], Theorem 2.17):

Proposition 2.2. *If e is an idempotent in a semigroup, then H_e is a group* □

Now a group contains only one idempotent, namely its identity, for if e is an idempotent in a group, then from the equation $e^2 = e$, it follows that e is the identity of the group, on multiplication by the inverse of e. This gives the following:

Corollary 2.1. *No two distinct idempotents in a semigroup are $\mathscr{H}$-related* □

In other words, if e is an idempotent in a semigroup, then e is the only idempotent in the $\mathscr{H}$-class H_e. On the other hand, the $\mathscr{L}$-class L_e and the $\mathscr{R}$-class R_e may possibly contain other idempotents. The next result gives a description of these idempotents. Note that for a subset X of a semigroup S, we denote by $\mathsf{E}(X)$, the set of all idempotents in S which belong to X.

Proposition 2.3. *Let e be an idempotent in a semigroup S. Then we have the following:*

(i) $\mathsf{E}(L_e) = \{f \in S : ef = e \text{ and } fe = f\}$

(ii) $\mathsf{E}(R_e) = \{f \in S : ef = f \text{ and } fe = e\}$

Proof. First let $f \in \mathsf{E}(L_e)$. Then $ef = e$, by Proposition 2.1. Also since $\mathscr{L}$ is a symmetric relation, we have $e \in \mathsf{E}(L_f)$, so that $fe = f$, by the same result. Conversely suppose that $f \in S$ with $ef = e$ and $fe = f$. Then $S^1e = S^1ef \subseteq S^1f$ and $S^1f = S^1fe \subseteq S^1e$, so that $S^1e = S^1f$, which means $e \mathrel{\mathscr{L}} f$. Hence $f \in L_e$. Moreover,

$$f^2 = (fe)(fe) = f(ef)e = fe^2 = fe = f$$

so that f is an idempotent. Thus $f \in \mathsf{E}(L_e)$. This proves (i). A dual argument proves (ii) □

Now for idempotents e and f in a semigroup, we have $e \mathrel{\mathscr{L}} f$ iff $f \in \mathsf{E}(L_e)$ and so by the above result, $ef = e$ and $fe = f$. Dually, $e \mathrel{\mathscr{R}} f$ iff $ef = f$ and $fe = e$. Thus we have the following:

Corollary 2.2. *Let e and f be idempotents in a semigroup. Then we have the following:*

(i) $e \mathrel{\mathscr{L}} f$ *if and only if* $ef = e$ *and* $fe = f$

(ii) $e \mathrel{\mathscr{R}} f$ *if and only if* $ef = f$ *and* $fe = e$ □

We next show that in the case (the multiplicative semigroup) of a ring, the set of idempotents in an $\mathscr{L}$-class or $\mathscr{R}$-class can be described in other ways. We begin with the simple observation below:

Lemma 2.1. *Let e and f be idempotents in a ring. If $e \mathrel{\mathscr{L}} f$ or $e \mathrel{\mathscr{R}} f$, then $(e-f)^2 = 0$*

Proof. By direct computation, we have

$$(e-f)^2 = e^2 - ef - fe + f^2 = e - ef - fe + f$$

If $e \mathrel{\mathscr{L}} f$, then $e - ef = 0 = f - fe$, so that $(e-f)^2 = 0$. If $e \mathrel{\mathscr{R}} f$, then $e - fe = 0 = f - ef$, so that again, $(e-f)^2 = 0$ □

Now an element x of a ring is said to be *nilpotent* if $x^n = 0$ for some natural number n and in this case, the least natural number n such that $x^n = 0$ is called the *index of nilpotence*. Thus the above result says that if e and f are distinct idempotents of a ring with $e \; (\mathscr{L} \bigcup \mathscr{R}) \; f$, then $e - f$ is a nilpotent of index 2.

Again if x is a nilpotent of index 2 in a ring R with unity (multiplicative identity) 1, then $(1-x)(1+x) = 1-x+x+x^2 = 1$ and similarly $(1+x)(1-x) = 1$.

Now an element u of a ring R with unity 1 is said to be a *unit*, if there exists an element u^{-1} of R such that $uu^{-1} = u^{-1}u = 1$. Thus the above argument shows that if x is a nilpotent of index 2 in R, then $1+x$ is a unit with $(1+x)^{-1} = 1-x$. This, together with the above proposition, gives the following:

Corollary 2.3. *Let e and f be idempotents in a ring R with unity 1. If $e \mathscr{L} f$ or $e \mathscr{R} f$, then $1+e-f$ is a unit in R with $(1+e-f)^{-1} = 1-e+f$* □

We can say more about the unit in the above result. Let e and f be idempotents in R with $e \mathscr{L} f$, so that $u = 1-e+f$ is a unit in R. Then

$$eu = e(1-e+f) = e - e^2 + ef = ef = e$$

and

$$ue = (1-e+f)e = e - e^2 + fe = fe = f$$

using Proposition 2.3. Thus $f = ue$ for a unit u in R which is a right identity for e. Conversely suppose $f = ue$ for a unit in R with $eu = e$. Then $fe = ue^2 = ue = f$ and $ef = e(ue) = (eu)e = e^2 = e$, so that $f \in \mathsf{E}(L_e)$, by the same proposition. Dually, we can show that $f \in \mathsf{E}(R_e)$ iff $f = eu$ for a unit in R which is left identity of e.

Note also that for an idempotent e in R, the set of all units which are right identities of e is a group; for if u and v are such units, then $e(uv) = (eu)v = ev = e$ and if u is any such unit, then $eu^{-1} = (eu)u^{-1} = e(uu^{-1}) = e$.

Thus we have proved the first part of the following result. The second part follows by dualizing the arguments.

Proposition 2.4. *Let R be a ring with unity and let G be the group of units in R. For each idempotent e in R, define*

$$G_e^r = \{u \in G \colon eu = e\} \;\; \text{and} \;\; G_e^l = \{u \in G \colon ue = e\}$$

Then G_e^r and G_e^l are subgroups of G with

$$\mathsf{E}(L_e) = G_e^r e \;\; \text{and} \;\; \mathsf{E}(R_e) = G_e^l e$$

for each idempotent e in R □

Another description of the set of idempotents in a $\mathscr{L}$-class or $\mathscr{R}$-class is given by the following (cf. [5], Part II, Chapter II, Lemma 2.7):

Proposition 2.5. *Let e be an idempotent in a ring R. Then we have the following:*

(i) $\mathsf{E}(L_e) = \{e + (1-e)xe \colon x \in R\}$

(ii) $\mathsf{E}(R_e) = \{e + ex(1-e) \colon x \in R\}$

Proof. First suppose that $f \in \mathsf{E}(L_e)$, so that $ef = e$ and $fe = f$. Let $x = f - e$, so that $f = e + x$. Then

$$ex = e(f-e) = ef - e^2 = e - e = 0$$

and

$$xe = (f-e)e = fe - e^2 = f - e = x$$

so that

$$(1-e)xe = xe - exe = x$$

and hence $f = e + x = e + (1-e)xe$.

Conversely, let $f = e + (1-e)xe$ for some x in R. Then

$$ef = e^2 + (e - e^2)xe = e$$

and

$$fe = e^2 + (1-e)xe^2 = e + (1-e)xe = f$$

So $f \in \mathsf{E}(L_e)$, by Proposition 2.3. This proves (i). A dual argument proves (ii) □

In any ring with unity, 0 and 1 are idempotents and it may happen that these are the only idempotents, as in the case of the ring of integers. We next give some sufficient conditions under which a good supply of idempotents is assured. For this we first note that if x is an element of a ring, we define nx for any natural number inductively as $2x = x + x$, $3x = 2x + x$ and so on. Also, for every natural number n, we define $(-n)x = n(-x)$ and $0a = 0$.

Now if e and f are idempotents in R with $e \mathscr{L} f$, then $f = e + (1-e)xe$ for some x in R, so that for each integer n,

$$(1-n)e + nf = (e - ne) + (ne + n(1-e)xe) = e + (1-e)(nx)e$$

and hence $(1-n)e + nf \in \mathsf{E}(L_e)$. Similarly, if $e \mathscr{R} f$, then $(1-n)e + nf \in \mathsf{E}(R_e)$ for each natural number n. Thus we have the following:

Lemma 2.2. *Let e and f be idempotents in a ring with $e\ (\mathscr{L} \bigcup \mathscr{R})\ f$ and for each natural number n, let $e_n = (1-n)e + nf$. Then e_n is an idempotent for each n. Moreover, if $e \mathscr{L} f$, then $e_n \in L_e$ and if $e \mathscr{R} f$, then $e_n \in R_e$ for each n* □

The idempotents e_n in the above result may not all be distinct, for if m and n are integers with $(m-n)(e-f) = 0$, then

$$e_m - e_n = ((1-m) - (1-n)e) + (m-n)f = (m-n)(f-e) = 0$$

so that $e_m = e_n$. So to ensure that the e_n are all distinct, it suffices to have $e \neq f$ and $nx \neq 0$ for all x in R and for all integers n.

An Abelian group G is said to be *torsion-free* if every $nx \neq 0$ for every $x \neq 0$ in G and for every natural number n. Since $(-n)x = n(-x)$ for every natural number n, it follows that $nx \neq 0$ for any $x \neq 0$ and for any non-zero integer n. So in a ring which is a torsion-free group under addition, if e and f are distinct idempotents which are $\mathscr{L}$-related or $\mathscr{R}$-related, then all the idempotents e_n are distinct.

For an idempotent e in the ring R, every idempotent $\mathscr{L}$-related to e is of the form $e + (xe - exe)$ and every idempotent $\mathscr{R}$-related to e is of the form $e + (ex - exe)$, by Proposition 2.5. Now if there exists x in R with $ex \neq xe$, then $ex - exe \neq xe - exe$, so that one of the elements $ex - exe$ or $xe - exe$ must be non-zero and hence one of the idempotents $e + xe - exe$ or $e + ex - exe$ is not equal to e.

An idempotent e in a ring R is said to be *central* if $ex = xe$ for all x in R. So, if R contains a non-central idempotent in R, then there exists x in R with $ex \neq xe$, so that either $e + xe - exe$ or $e + ex - exe$ is an idempotent not equal to e. If f is this idempotent, then $f\ (\mathscr{L} \bigcup \mathscr{R})\ e$ and $f \neq e$. So, if we further assume that the additive group R is torsion-free, then none of the idempotents e_n are equal and so gives an infinite set of idempotents. Thus we have the following:

Proposition 2.6. *Let R be a ring which is a torsion-free group under addition. If R contains a non-central idempotent, then it contains infinitely many idempotents* □

Now for any idempotent e in a ring R with unity, the element $1 - e$ is also an idempotent, since

$$(1 - e)^2 = 1 - e - e + e^2 = 1 - 2e + e = 1 - e$$

Moreover $e \neq 1 - e$, for if $e = 1 - e$, then $e = ee = e(1 - e) = 0$ and also $1 - e = (1 - e)(1 - e) = e(1 - e) = 0$ which gives $e = 1$. So idempotents in a regular ring occur in distinct pairs $(e, 1 - e)$. Also, $1 - e = 1 - f$ iff $e = f$. This gives the following:

Proposition 2.7. *If a ring with unity has only a finite number of idempotents, then this number is even* □

3 Orthogonal idempotents

We have noted that in a ring with unity, each idempotent has an associated idempotent $1 - e$. Also, we can see that the map $e \mapsto (1 - e)$ dualizes and reverses certain relations between idempotents. To make this precise, we split each of the relations $\mathscr{L}$ and $\mathscr{R}$ on the set of idempotents in a semigroup S as the intersection of two relations. Using the notations used in [4], we define for idempotents e and f in S,

$$e\ \omega^l\ f \text{ iff } S^1e \subseteq S^1f$$
$$e\ \omega^r\ f \text{ iff } eS^1 \subseteq fS^1$$

Now if $S^1e \subseteq S^1f$, then $e \in S^1f$ so that $e = xf$ for some x in S, so that $ef = xf^2 = xf = e$. Conversely if $ef = e$, then $S^1e = S^1ef \subseteq S^1f$. Similarly, $eS^1 \subseteq fS^1$ iff $fe = e$. Thus the relations above can be described without reference to ideals as follows:

Proposition 3.1. *Let e and f be idempotents in a semigroup. Then we have the following:*

(i) $e\ \omega^l\ f$ *if and only if* $ef = e$

(ii) $e\ \omega^r\ f$ *if and only if* $fe = e$ □

Note that if $e\ \omega^l\ f$, then e satisfies half the conditions for being $\mathscr{L}$-related to f and being ω-related to f. We next show that in this case, the product fe is both $\mathscr{L}$-related to f and ω-related to f (cf. [4], Theorem 1.1):

Proposition 3.2. *For idempotents e and f of a semigroup,*

(i) *if $e\ \omega^l\ f$, then fe is an idempotent with $fe\ \mathscr{L}\ e$ and $fe\ \omega\ f$*

(ii) *if $e\ \omega^r\ f$, then ef is an idempotent with $ef\ \mathscr{R}\ e$ and $ef\ \omega\ f$* □

Proof. First note that for idempotents e and f of a semigroup, if $e\ \omega^l\ f$, then $ef = e$, by definition, so that

$$(fe)(fe) = f(ef)e = fee = fe$$

and so fe is an idempotent. Also $(fe)e = fe$ and $e(fe) = (ef)e = e$, so that $fe\ \mathscr{L}\ e$; and $(fe)f = f(ef) = fe$ and $f(fe) = fe$, so that $fe\ \omega\ f$. Dually, we can show that if $e\ \omega^r\ f$, then ef is an idempotent with $ef\ \mathscr{R}\ e$ and $ef\ \omega\ f$ □

Now we clarify our earlier statement about the map $e \mapsto (1-e)$ in a ring with unity (cf.[3], Corollary 2.2.5). Throughout the following R denotes a ring with unity 1, unless otherwise specified.

Proposition 3.3. *Let e and f be idempotents in R. Then we have the following:*

(i) $e\ \omega^l\ f$ *if and only if* $(1-f)\ \omega^r\ (1-e)$

(ii) $e\,\omega^r\,f$ *if and only if* $(1-f)\,\omega^l\,(1-e)$

Proof. First let $e\,\omega^l\,f$, so that $ef = e$ and so

$$(1-e)(1-f) = 1 - e - f + ef = 1 - f$$

which means $(1-f)\,\omega^r\,(1-e)$. Again, if $e\,\omega^r\,f$, then $fe = e$, so that

$$(1-f)(1-e) = 1 - f - e + fe = 1 - f$$

and so $(1-f)\,\omega^l\,(1-e)$. The reverse implications follow from the fact that $1-(1-e) = e$ for every idempotent e in R □

Using the fact that $\mathscr{L} = \omega^l \cap (\omega^l)^{-1}$ and $\mathscr{R} = \omega^r \cap (\omega^r)^{-1}$, we get our next result.

Corollary 3.1. *Let e and f be idempotents in R. Then we have the following:*

(i) $e\,\mathscr{L}\,f$ *iff* $(1-e)\,\mathscr{R}\,(1-f)$

(ii) $e\,\mathscr{R}\,f$ *iff* $(1-e)\,\mathscr{L}\,(1-f)$ □

Note that $e(1-e) = (1-e)e = 0$, for each idempotent e. In the following, an idempotent e of R is said to be *orthogonal* to an idempotent f, written $e \perp f$, if $ef = fe = 0$. Thus for each idempotent e of R, we have $(1-e) \perp e$.

The relations between e and $1-e$ can be slightly generalized. For this, we first define the relation

$$\omega = \omega^l \cap \omega^r$$

on the idempotents in a semigroup, as in [4]. Since ω^l and ω^r are easily seen to reflexive and transitive, so is ω. Suppose $e\ \omega\ f$ and $f\ \omega\ e$. Then $ef = e$, since $e\,\omega^l\,f$ and $ef = f$, since $f\,\omega^r\,e$, by Proposition 3.1, so that $e = ef = f$. Thus ω is a partial order on the set of idempotents in S.

Now in a semigroup with multiplicative identity 1, we have $e\ \omega\ 1$ for every idempotent e. Thus for each idempotent e of R we have $e\ \omega\ 1$ and $(1-e) \perp e$. This can be generalized as follows:

Proposition 3.4. *If e and f are idempotents in R with $e\ \omega\ f$, then $f-e$ is an idempotent in R with $(f-e)\ \omega\ f$ and $(f-e) \perp e$* □

Proof. Let e and f be idempotents in R with $e\ \omega\ f$, so that $ef = fe = e$. Then

$$(f-e)^2 = f - fe - ef + e = f - e - e + e = f - e$$

so that $f-e$ is an idempotent. Also,

$$f(f-e) = f - fe = f - e$$

and

$$(f - e)f = f - ef = f - e$$

so that $(f - e)\ \omega\ f$. Again,

$$e(f - e) = ef - e = 0$$

and

$$(f - e)e = fe - e = 0$$

so that $f - e$ is orthogonal to e □

We can also generalize part of Proposition 3.3 as follows:

Proposition 3.5. *Let e and f be idempotents in R and let $e\ \omega\ g$ and $f\ \omega\ g$ for some idempotent g in R. Then we have the following:*

(i) *If $e\ \omega^l\ f$, then $(g - f)\ \omega^r\ (g - e)$*

(ii) *If $e\ \omega^r\ f$, then $(g - f)\ \omega^l\ (g - e)$*

Proof. Since $e\ \omega\ g$ and $f\ \omega\ g$, the elements $g - e$ and $g - f$ are idempotents, by Proposition 3.3. Now if $e\ \omega^l\ f$, then $ef = e$, so that

$$(g - e)(g - f) = g - gf - eg + ef = g - f - e + e = g - f$$

since e and f are ω-related to g. Hence $(g - f)\ \omega^r\ (g - e)$. Similarly we can show that if $e\ \omega^r\ f$, then $(g - f)\ \omega^l\ (g - e)$ □

Note also that in Proposition 3.4, we have $f = e + (f - e)$ so that f is decomposed as the sum of a pair orthogonal idempotents smaller than f with respect to the partial order ω. The next result shows that the sum of any pair of orthogonal idempotents gives a larger idempotent.

Proposition 3.6. *If e and f are idempotents in R with $e \perp f$, then $e + f$ is an idempotent with $e\ \omega\ (e + f)$ and $f\ \omega\ (e + f)$*

Proof. Direct computation gives

$$(e + f)^2 = (e + f)(e + f) = e + ef + fe + f$$

so that if $ef = fe = 0$, then $e+f$ is also an idempotent. Also, $e(e+f) = e+ef = e$ and $(e + f)e = e + fe = e$, so that $e\ \omega\ (e + f)$ and $f(e + f) = fe + f = f$ and $(e + f)f = ef + f = f$, so that $f\ \omega\ (e + f)$ □

Now if e, f, g are pairwise orthogonal idempotents in R, then $e + f$ is an idempotent by this result, and $(e + f)g = 0 = g(e + f)$, since g is orthogonal to e and f. Thus $(e + f) \perp g$ and so $e + f + g$ is an idempotent. Again by orthogonality, it follows that e, f, g are all ω-related to $e + f + g$. By induction, we have the following:

Corollary 3.2. *If $\{e_1, e_2, \ldots, e_n\}$ is a finite set of pairwise orthogonal idempotents in R, then $e = e_1 + e_2 + \cdots + e_n$ is an idempotent with $e_k \;\omega\; e$ for $k = 1, 2, \ldots, n$* □

We next note that either of the two equations defining orthogonality can be characterized in terms of the relations ω^l and ω^r:

Proposition 3.7. *Let e and f be idempotents in R. Then the following are equivalent:*

(i) $ef = 0$

(ii) $e\,\omega^l\,(1 - f)$

(iii) $f\,\omega^r\,(1 - e)$

Proof. If $ef = 0$, then $e(1 - f) = e - ef = e$. This prove that (i) implies (ii). From Proposition 3.3, it follows that (ii) implies (iii). Finally if $f\,\omega^r\,(1 - e)$, then $(1 - e)f = f$ from which we get $ef = 0$. Thus (iii) implies (i) □

The following characterization of orthogonality is immediate:

Corollary 3.3. *Let e and f be idempotents in R. Then the following are equivalent:*

(i) $e \perp f$

(ii) $e\,\omega\,(1 - f)$

(iii) $f\,\omega\,(1 - e)$ □

4 Conclusions

Idempotents of semigroups play an important role in the structure of semigroups. Every ring is a semigroup under multiplication. In this paper the properties of idempotents in a ring are studied by viewing ring as a semigroup.

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